Pushing the Bounds of Subtyping

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Some ideas are based on joint works with Julien Cretin and Gabriel Scherer

INRIA

Luca Cardelli’s Fest

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The dawn of subtyping

Early ideas in the 60’s. First formalization by Reynolds (1980).
The Amber language (Cardelli, 1984)
Type system based on simply-typed $\lambda$-calculus
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Type system based on simply-typed $\lambda$-calculus + a subtyping rule:

$$\Gamma \vdash a : \tau \quad \tau \leq \tau'$$

$$\Gamma \vdash a : \tau'$$
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\]

Subtyping is defined by:

\[
\perp \leq \tau \quad \tau \leq \top \quad \frac{\tau_1' \leq \tau_1 \quad \tau_2 \leq \tau_2'}{(\tau_1 \rightarrow \tau_2) \leq (\tau_1' \rightarrow \tau_2')}
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Amber also had records and subtyping on records that were at the basis of object encodings and inheritance.

and many more features:

- recursive types
- dynamic types
- concurrency primitives
- modules

Each of which later became one of Luca’s research topic
Bounded quantification

How to mix subtyping with polymorphism?

∀(α) σ

System F
How to mix subtyping with polymorphism: Cardelli and Wegner (1985)

\[ \forall (\alpha \leq \tau) \sigma \]

Quest, \( F < \)
Bounded quantification

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\[ \Gamma, \alpha \leq \tau \vdash \sigma \leq \sigma' \]
Bounded quantification

How to mix subtyping with polymorphism: Cardelli and Wegner (1985)

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\Gamma, \alpha \leq \tau \vdash \sigma \leq \sigma'
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\[
\Gamma \vdash \forall (\alpha \leq \tau) \sigma \leq \forall (\alpha \leq \tau) \sigma'
\]
Bounded quantification

How to mix subtyping with polymorphism: Cardelli and Wegner (1985)

\[ \Gamma, \alpha \leq \tau \vdash \sigma \leq \sigma' \]

\[ \Gamma \vdash \forall (\alpha \leq \tau) \sigma \leq \forall (\alpha \leq \tau) \sigma' \]

Bounded quantification is quite expressive and at the basis of most works on type systems in the 1990’s, including *A Theory of Objects.*
Bounded quantification

How to mix subtyping with polymorphism: Cardelli and Wegner (1985)

\[ \Gamma \vdash \forall (\alpha \leq \tau) \sigma \not\leq \sigma[\alpha \leftarrow \tau'] \]

Still, bounded quantification is somewhat limited

- to a single, upper bound
- and does not allow instantiation of quantifiers.
Bounded quantification

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Still, bounded quantification is somewhat limited

- to a single, upper bound
- and does not allow instantiation of quantifiers.

This keeps subtyping decidable* and trackable:

\[ \forall \text{ is explicit, so that } \leq \text{ can be left implicit.} \]
Type instantiation as subtyping

In ML

$$\forall (\alpha) \sigma \leq \sigma[\alpha \leftarrow \tau]$$

Also used in type containment (John Mitchell, 1984):
- mixes type instantiation with the Amber rules, but
- does not allow reasoning under subtyping assumptions
Instance-bounded quantification

Introduced for partial type inference in MLF (Le Botlan & Rémy, 2003)

\[ \forall (\alpha \geq \tau) \sigma \]

Stands for the set of types \( \sigma \) where \( \alpha \) ranges over instances of \( \tau \).

This avoids having to decide too early whether types should be instantiated or kept polymorphic.
Can all features be combined together?

e.g. MLF + subtyping? $F_\leq +$ Type containment?

or

Restrict subtyping to equivalences? (as in the internal language of Haskell)
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Restrict subtyping to equivalences?
(as in the internal language of Haskell)

In fact, we also need...

- multiple bounds, *e.g.* as in ML with subtyping constraints.
- general, equi-recursive types and coinduction.
Full reduction semantics

Reduction should be allowed in any context:
- for our understanding, because it should be sound to do so.
- this models reduction of open terms
- this avoids postponing type errors to type-instantiation sites
- soundness will remain true for any strategy

I also argue that
- full reduction is a better, more abstract model for the user
- CBV is a more concrete model, only needed to reason about costs
Subtyping as constrained kinds

\[ \tau ::= \ldots \ | \ \forall (\alpha \mid P) \tau \quad \quad \quad P ::= \tau \leq \tau \ | \ P \land P \ | \ldots \]

Intuitively, introduce constrained quantification
Subtyping as constrained kinds

$$\tau ::= \ldots \mid \forall (\alpha : k) \tau$$

$$P ::= \tau \leq \tau \mid P \land P \mid \ldots$$

$$k ::= \star \mid \{\alpha \mid P\} \mid \ldots$$

More conveniently, using kinds... similar to power kinds (Cardelli, 1988)
Subtyping as constrained kinds

\[ \tau ::= \ldots | \forall (\alpha : k) \, \tau \]
\[ P ::= \tau \leq \tau | P \land P | \ldots \]
\[ k ::= \star | \{\alpha | P\} | \ldots \]

Coherence

Kinds must be inhabited. Otherwise, type errors could be hidden behind abstraction over the absurd:

\[ \Gamma, \alpha : \{\beta : \text{int} \leq \text{bool} \to \text{int}\} \vdash \text{int} \leq \text{bool} \to \text{int} \]

\[ \vdash \]

\[ \Gamma, \alpha : \{\beta : \text{int} \leq \text{bool} \to \text{int}\} \vdash 1 \, \text{true} : \text{int} \]

\[ \Gamma \vdash 1 \, \text{true} : \forall (\alpha : \{\beta | \text{int} \leq \text{bool} \to \text{int}\}) \, \text{int} \]

Terms with incoherent constraints cannot be (safely) reduced.
Subtyping as constrained kinds

\[ \tau ::= \ldots \mid \forall (\alpha : k) \tau \]

\[ P ::= \tau \leq \tau \mid P \land P \mid \ldots \]

\[ k ::= \star \mid \{ \alpha \mid P \} \mid \ldots \]

Coherence

Kinds must be inhabited.

\[ \Gamma, \alpha : \kappa \vdash \tau : \star \quad \Gamma \vdash \sigma : \kappa \]

\[ \Gamma \vdash \forall (\alpha : k) \tau : \star \]
Incoherence

Incoherence is also useful

- A kind may not be inhabited for all but only some instances of the current context—a typical situation with GADTs.
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$$\forall^\dagger (\alpha : k) \, \tau$$
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We allow a distinct form of incoherent type abstraction

\[ \forall^\dagger(\alpha : k) \, \tau \quad \text{a ::= ... } \mid \partial a \]

- Delay evaluation at the introduction of incoherent type abstraction.

\[
\frac{\Gamma, \alpha : \kappa \vdash a : \tau \quad \Gamma \vdash \sigma : k}{\Gamma \vdash \partial a : \forall^\dagger(\alpha : k) \, \tau}
\]
**Incoherence**

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**We allow a distinct form of incoherent type abstraction**

\[ \forall^{\dagger}(\alpha : k) \tau \]

\[ a ::= \ldots \mid \partial a \mid a \diamond \]

- Delay evaluation at the introduction of incoherent type abstraction.
- Resume evaluation at its elimination.

\[
\begin{align*}
\Gamma, \alpha : \kappa & \vdash a : \tau & \Gamma \vdash \sigma : k \\
\Gamma \vdash \partial a : \forall^{\dagger}(\alpha : k) \tau \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \partial a : \forall^{\dagger}(\alpha : k) \tau & \quad \Gamma \vdash \sigma : k \\
\Gamma, \alpha : \kappa & \vdash a \diamond : \tau[\alpha \leftarrow \sigma] \\
\end{align*}
\]
To conclude our short journey

We have an external language for exploring the design space

- Our type system is sound,

- but subject reduction does not hold.

- Thus it is not quite suitable for

  - an internal language (by lack of subject reduction)
  - nor for a surface language
    (an explicitly-typed version would be very verbose)

- But it does help share and separate

  - the meta-theoretical study (what can be done, safely)
  - from the practical design (what restrictions should be made)
Subtyping played an important role in Luca’s early research. Luca contributed a lot to make subtyping a well-understood feature. Still, he left us a few variants and new uses of subtyping to explore.
Thank you Luca for all your inspiring works and a profusion of challenging new ideas that always kept us moving forward, faster and further.