Finite Developments in the λ-calculus

Part 3

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Inside-out reductions

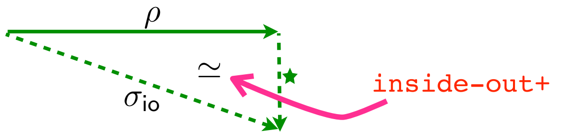
- **Definition:** The following reduction is **inside-out**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j inside R_i in M_{i-1}.

- **Theorem** [Inside-out completeness, 74]

Let $M \xrightarrow{*} N$. Then $M \xrightarrow[\text{io}]{*} P$ and $N \xrightarrow{*} P$ for some P.



Labeled lambda-calculus

Exercise 1 Show that residuals of redexes keep same names by case inspection on occurrences of redexes.

Exercise 2 Show that $M \longrightarrow N$ implies $M^\alpha \longrightarrow N^\alpha$

Exercise 3 Show the parallel moves lemma (with Martin-Löf way)

If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.

Exercise 4 Label Y_f, draw its reduction graph and show redexes families when $Y_f = (\lambda x.f(xx))(\lambda x.f(xx))$

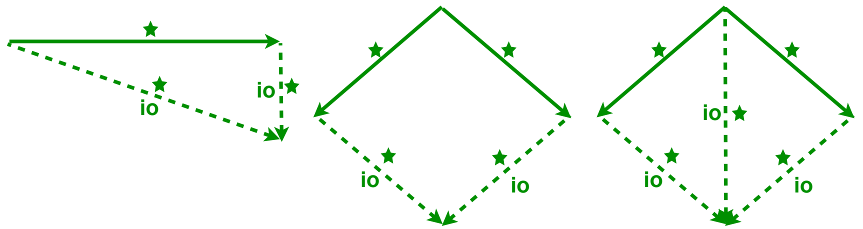
Exercise 5 Same with KaY_f

Exercises

Exercise 6 Prove inside-out completeness

Hint: use Finite Development theorem.

Exercise 7 Prove the following diagrams

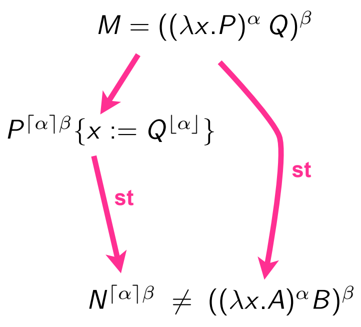


Permutation equivalence

Proof [uniqueness of labeled standard]

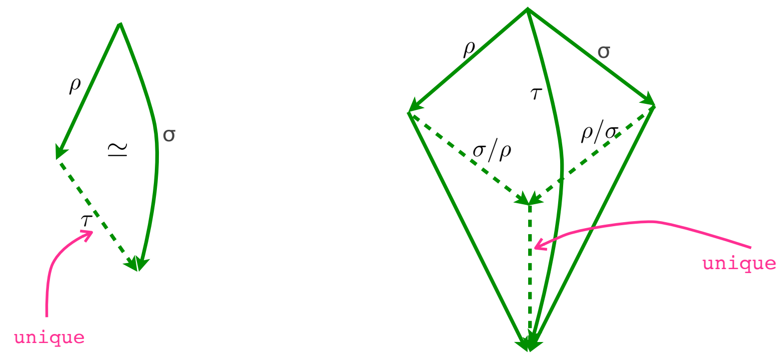
Let ρ and σ be 2 distinct coinitial pure labeled standard reductions.

Take first step when they diverge. Call M that term.
We make structural induction on M. Say ρ is more to the left.
If first step of ρ contracts an internal redex, we use induction.
If first step of ρ contracts an external redex, then:



Permutation equivalence

- **Exercise 9** Show the following diagrams



Permutation equivalence

- **Corollary** [labeled prefix ordering]
  Let $\rho : M \xrightarrow{*} N$ and $\sigma : M \xrightarrow{*} P$ be coinitial pure labeled reductions.
  Then $\rho \sqsubseteq \sigma$ iff $N \xrightarrow{*} P$.

- **Exercise 8** Show the following properties

  (i) $\rho \sqsubseteq \rho$

  (ii) $\rho \sqsubseteq \sigma \sqsubseteq \rho$ implies $\rho \simeq \sigma$

  (iii) $\rho \sqsubseteq \sigma \sqsubseteq \tau$ implies $\rho \simeq \tau$

  (iv) $\rho \sqsubseteq \sigma$ implies $\rho/\tau \sqsubseteq \sigma/\tau$

  (v) $\rho \sqsubseteq \sigma$ iff $\exists \tau,\ \rho\tau \simeq \sigma$

  (vi) $\rho \sqsubseteq \rho \sqcup \sigma,\ \sigma \sqsubseteq \rho \sqcup \sigma$

  (vii) $\rho \sqsubseteq \tau,\ \sigma \sqsubseteq \tau$ implies $\rho \sqcup \sigma \sqsubseteq \tau$

Permutation equivalence

- **Corollary** [lattice of labeled reductions]
  Labeled reduction graphs are upwards semi lattices for any pure labeling.

- **Corollary** [push-out category]
  Prefix ordering on reductions is a push-out.

- **Exercise 10** Try on $(\lambda x.x)((\lambda y.(\lambda x.x)a)b)$ or $(\lambda x.xx)(\lambda x.xx)$

- **Exercise 11** Show that prefix ordering on reductions is not a pull-back.

Proof of the GFD theorem

Proof of finite developments

- **Lemma 1** Let $M \xrightarrow{*} N$, then $h(\tau(M)) \leq h(\tau(N))$

Proof by induction on length of reduction. Let $M \xrightarrow{R} N,\ R = ((\lambda x.A)^\alpha B)^\beta$
If R is internal in M, then $\tau(M) = \tau(N)$.
If $M = R = ((\lambda x.A)^\alpha B)^\beta \longrightarrow A\{x := B^{\lfloor \alpha \rfloor}\}^{\lceil \alpha \rceil \beta} = N$,
then $h(\tau(M)) = h(\beta) \leq h(\gamma\beta) = h(\tau(N))$ for some γ.

- **Lemma 2** Let $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \xrightarrow{*} (\lambda x.N)^\alpha$
  Then $h(\tau(M)) \leq h(\alpha)$

Proof by induction on n.
When $n = 0$, obvious by lemma 1.
Otherwise $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_{n-1})^{\beta_{n-1}} \xrightarrow{*} (\lambda y.P)^\gamma$
and $((\lambda y.P)^\gamma Q)^{\beta_n} \longrightarrow P\{y := Q^{\lfloor \gamma \rfloor}\}^{\lceil \gamma \rceil \beta_n} \xrightarrow{*} (\lambda x.N)^\alpha$
So $h(\tau(M)) \leq h(\gamma) < h(\delta\lceil \gamma \rceil \beta_n) \leq h(\alpha)$ by induction and lemma 1.

Proof of finite developments

- **Notation** $\tau(M^\alpha) = \alpha$ when M has an empty external label

- **Lemma 1** Let $M \xrightarrow{*} M'$, then $h(\tau(M)) \leq h(\tau(M'))$

- **Lemma 2** Let $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \xrightarrow{*} (\lambda x.N)^\alpha$
  Then $h(\tau(M)) \leq h(\alpha)$

- **Lemma 3** [Barendregt] Let $M\{x := N\} \xrightarrow{*} (\lambda y.P)^\alpha$
  There are 2 cases:
  $M \xrightarrow{*} (\lambda y.M')^\alpha$ and $M'\{x := N\} \xrightarrow{*} P$
  $M \xrightarrow{*} M' = (\cdots((x^\beta\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n}$ and $M'\{x := N\} \xrightarrow{*} (\lambda y.P)^\alpha$

Proof of finite developments

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Proof Let $M^* = M\{x := N\}$. There are 3 cases on weak head reduction of M :
it reaches an abstraction or a head variable which has to be x.
More precisely, we consider the standard reduction from M^* to $(\lambda y.P)^\alpha$.

Case 1: $M = (\lambda y.M')^\alpha$ and we are done since $M^* = (\lambda y.M'^*)^\alpha$.

Case 2: $M = ((\cdots((y^\beta\, M_1)^{\beta_1}\, M_2)^{\beta_2}) \cdots M_n)^{\beta_n}$. Then $y = x$ and $M' = M$.

Case 3: $M = (\cdots((((\lambda z.A)^\beta\, B)^\gamma\, C_1)^{\beta_1}\, C_2)^{\beta_2} \cdots C_n)^{\beta_n}$
Let $M_1 = (\cdots((A\{z := B^{\lfloor \beta \rfloor}\}^{\lceil \beta \rceil \gamma}\, C_1)^{\beta_1}\, C_2)^{\beta_2} \cdots C_n)^{\beta_n}$
Then $M^* = (\cdots((((\lambda z.A^*)^\beta\, B^*)^{\beta_1}\, C_1^*)^{\beta_1}\, C_2^*)^{\beta_2} \cdots C_n^*)^{\beta_n} \longrightarrow M_1^*$ is the first step of the standard reduction from M^* to $(\lambda y.P)^\alpha$. By induction on its length, we are done.

Proof of finite developments

- **Notation** Let $\mathcal{SN}_{\mathcal{F}}$ be the set of strongly normalizable terms w.r.t. reductions relative to $\mathcal{F}$.

- **Lemma** `[subst]` Let $\mathcal{F}$ be a finite set of redex families.
  $M, N \in \mathcal{SN}_{\mathcal{F}}$ implies $M\{x := N\} \in \mathcal{SN}_{\mathcal{F}}$

  Proof `[van Daalen]` by induction on $\langle H(\mathcal{F}) - h(\tau(N)), \text{depth}(M), \|M\| \rangle$

- **Theorem** `GFD` Let $\mathcal{F}$ be a finite set of redex families.
  Then $M \in \mathcal{SN}_{\mathcal{F}}$ for all M.

  Proof by easy induction on $\|M\|$

Proof of finite developments

- **Lemma** `[subst]` Let $\mathcal{F}$ be a finite set of redex families.
  $M, N \in \mathcal{SN}_{\mathcal{F}}$ implies $M\{x := N\} \in \mathcal{SN}_{\mathcal{F}}$

Proof `[van Daalen]` by induction on $\langle H(\mathcal{F}) - h(\tau(N)), \text{depth}(M), \|M\| \rangle$

Cases $M = x$, $M = y$, $M = \lambda y.M_1$ are obvious or easy by induction on $\|M\|$.

Write M^* for $M\{x := N\}$ and consider case $M = (M_1 M_2)^\alpha$.

If all reductions are internal to M_1^* and M_2^*, then easy induction on $\|M\|$.

Otherwise, let $M_1^* \xrightarrow{*} (\lambda y.P)^\beta$ and $M_2^* \xrightarrow{*} Q$ and $((\lambda y.P)^\beta Q)^\alpha \longrightarrow P\{y := Q^{\{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$

Then M_1^* and M_2^* are in $\mathcal{SN}_{\mathcal{F}}$ by induction on $\|M\|$,

and $M_1^* \xrightarrow{*} (\lambda y.P)^\beta$ and $M_2^* \xrightarrow{*} Q$. So P and Q are in $\mathcal{SN}_{\mathcal{F}}$.

How is $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$??

By lemma 3, we have 2 cases:

Proof of finite developments

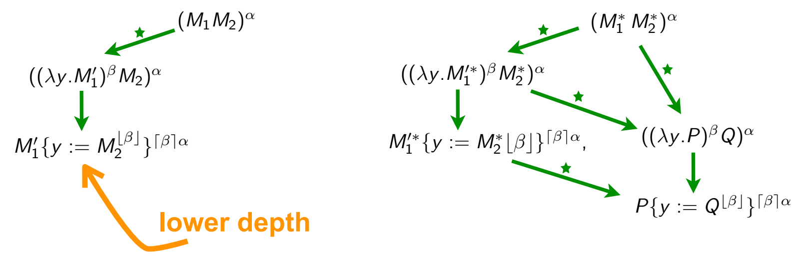
Case 1:

Then $M_1 \xrightarrow{*} (\lambda y.M_1')^\beta$ and $M_1'^* \xrightarrow{*} P$.

Therefore $M_1'^*\{y := M_2^{*\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \xrightarrow{*} P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$.

But as $M = (M_1 M_2)^\alpha \xrightarrow{*} ((\lambda y.M_1')^\beta M_2)^\alpha \longrightarrow M' = M_1'\{y := M_2^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$,
we have $\text{depth}(M') < \text{depth}(M)$.

Thus by induction $M'^* = M_1'^*\{y := M_2^{*\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_{\mathcal{F}}$
and $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_{\mathcal{F}}$.



Proof of finite developments

Case 2:

$M_1 \xrightarrow{*} M_1' = (\cdots((x^\gamma A_1)^{\gamma_1} A_2)^{\gamma_2})\cdots A_n)^{\gamma_n}$ and
$M_1'^* = (\cdots((N^\gamma A_1^*)^{\gamma_1} A_2^*)^{\gamma_2})\cdots A_n^*)^{\gamma_n} \xrightarrow{*} (\lambda y.P)^\beta$

Therefore $h(\tau(N)) \le h(\tau(N^\gamma)) \le h(\beta)$ by lemma 2.

So $M^* = (M_1^* M_2^*)^\alpha \xrightarrow{*} ((\lambda y.P)^\alpha)Q)^\beta \xrightarrow{*} P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$

and $h(\tau(N)) \le h(\beta) < h(\lfloor\beta\rfloor) \le h(\tau(Q^{\lfloor\beta\rfloor}))$.

We get by induction $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_{\mathcal{F}}$.