Comparing a Formal Proof

in Why3, Coq, Isabelle

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Motivation

- nice algorithms should have simple formal proofs
- to be fully published in articles or journals
- how to publish formal proofs ?
- algorithms on graphs = a good testbed (better than $\sqrt{2}$)
- formal proofs have to be checked by computer

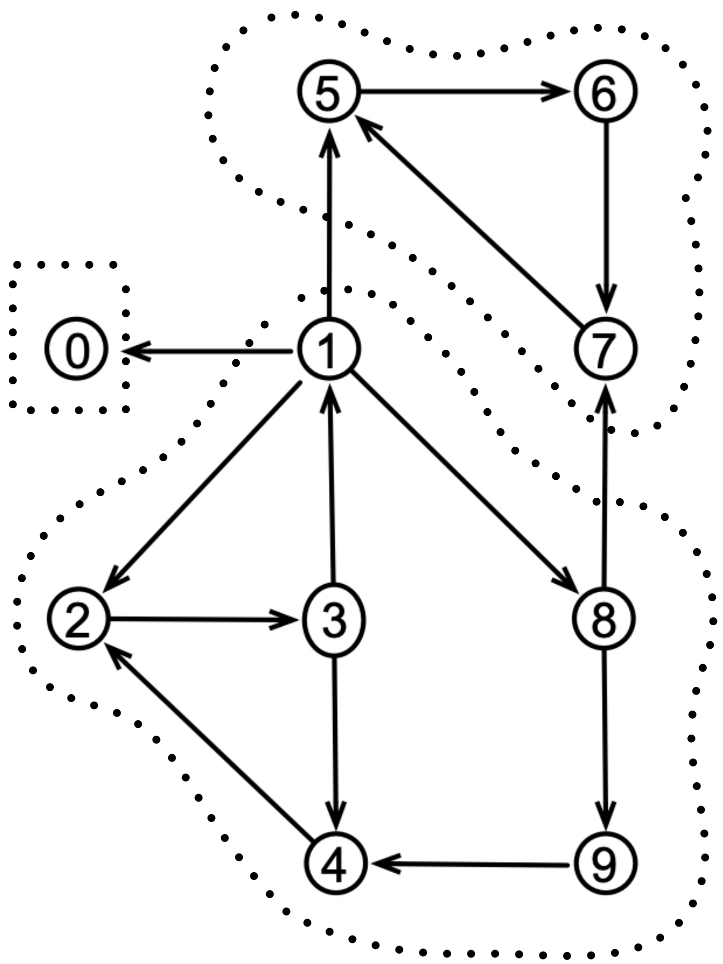
.. with Ran Chen, Cyril Cohen, Stephan Merz, Laurent Théry VSTTE 2017, ITP 2019

`http://www-sop.inria.fr/marelle/Tarjan/contributions.html`

A one-pass linear-time algorithm

Tarjan, 1972

Strongly connected components



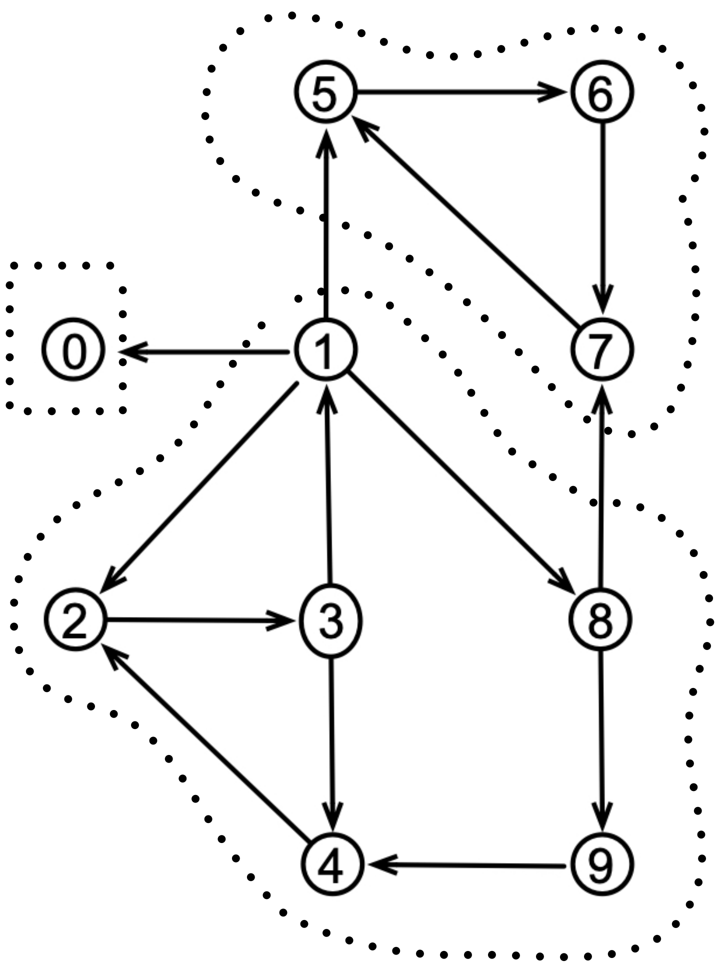
- x and y are strongly connected if there exists a path from x to y and a path from y to x

- scc is a maximal set of vertices in which each pair is strongly connected

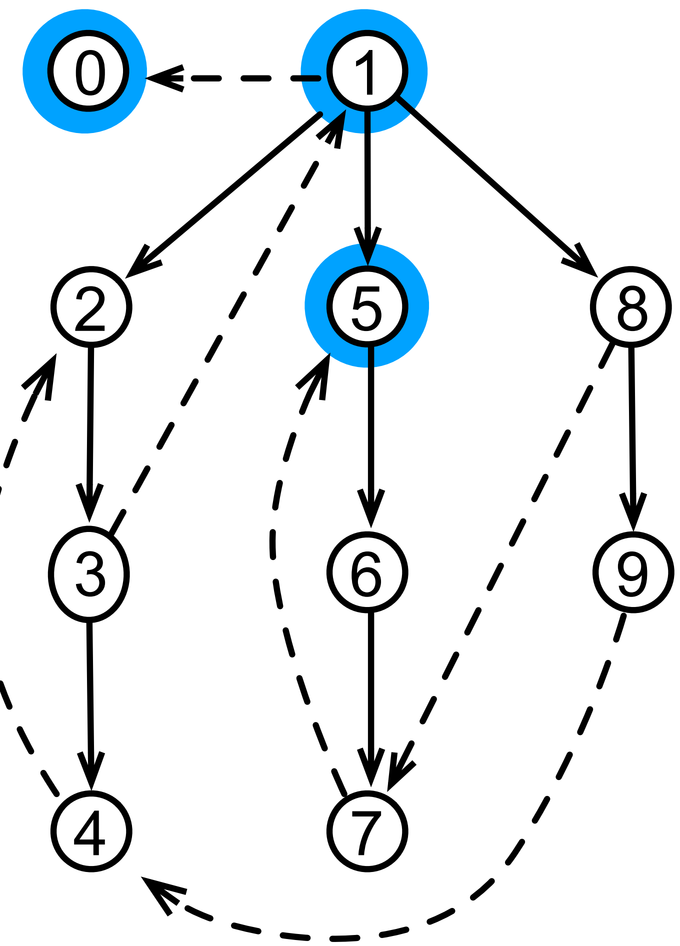
- depth-first search algorithm tracking **bases** of scc's

- vertices are pushed on a stack in order of their visit and popped when the **base** of a scc is found

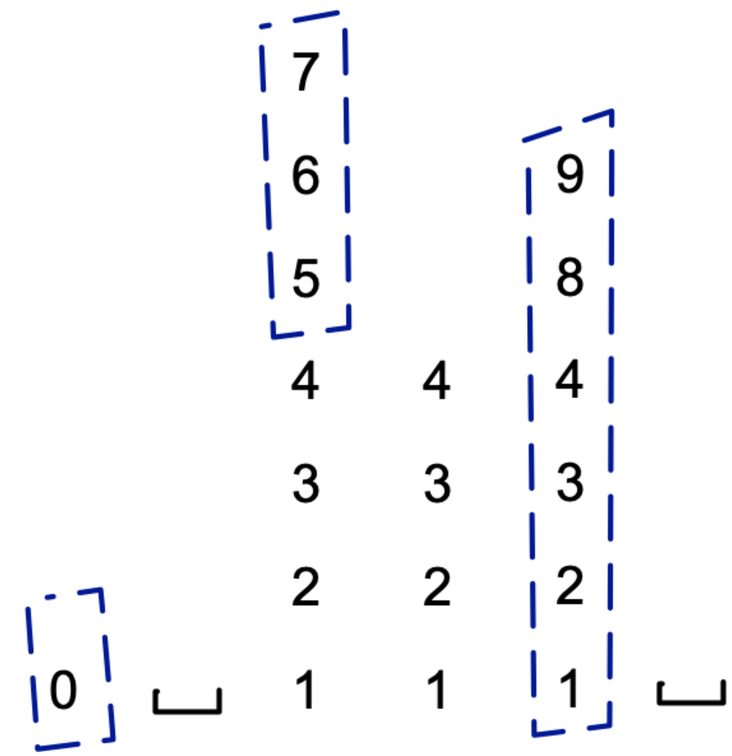
Strongly connected components



graph



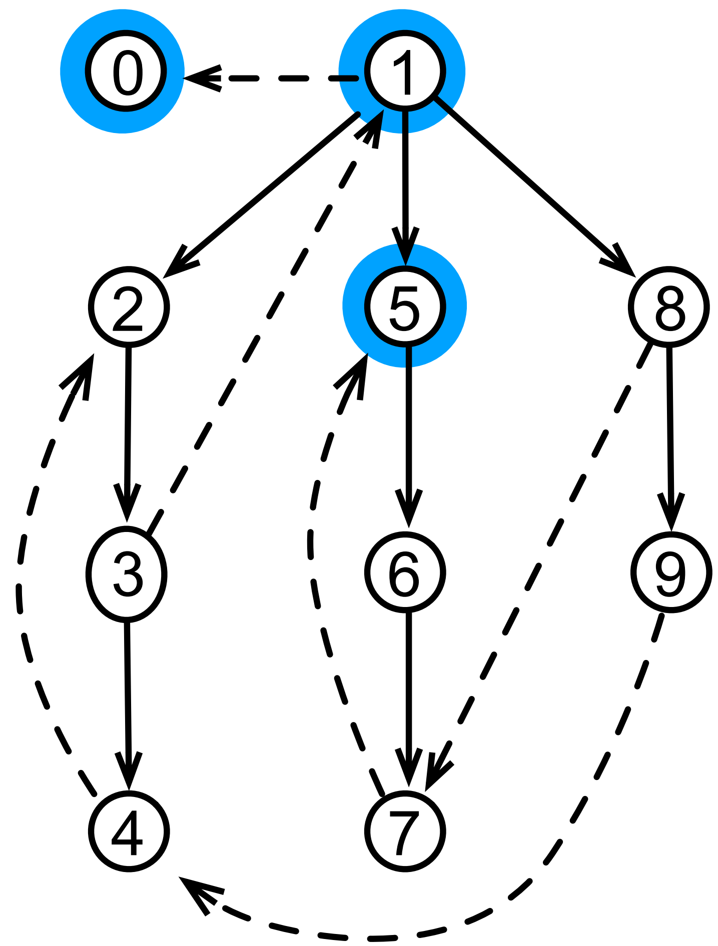
spanning forrest



stack

Strongly connected components

0	$+\infty$
1	1
2	1
3	1
4	2
5	5
6	5
7	5
8	4
9	4



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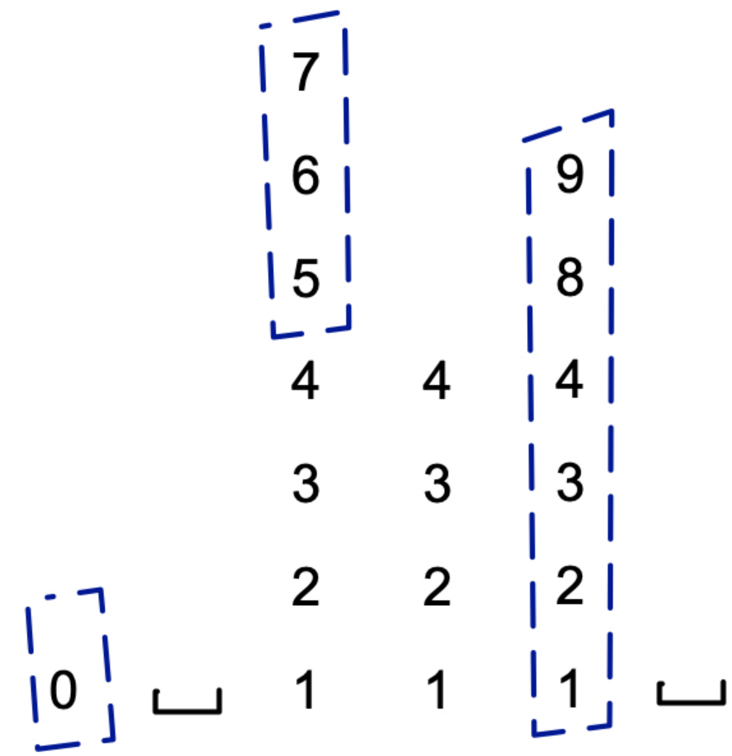


stack

$$LOWLINK(x) = \min\{num[y] \mid x \overset{*}{\Longrightarrow} z \hookrightarrow y$$
$$\wedge\ x \text{ and } y \text{ are in the same connected component}\}$$

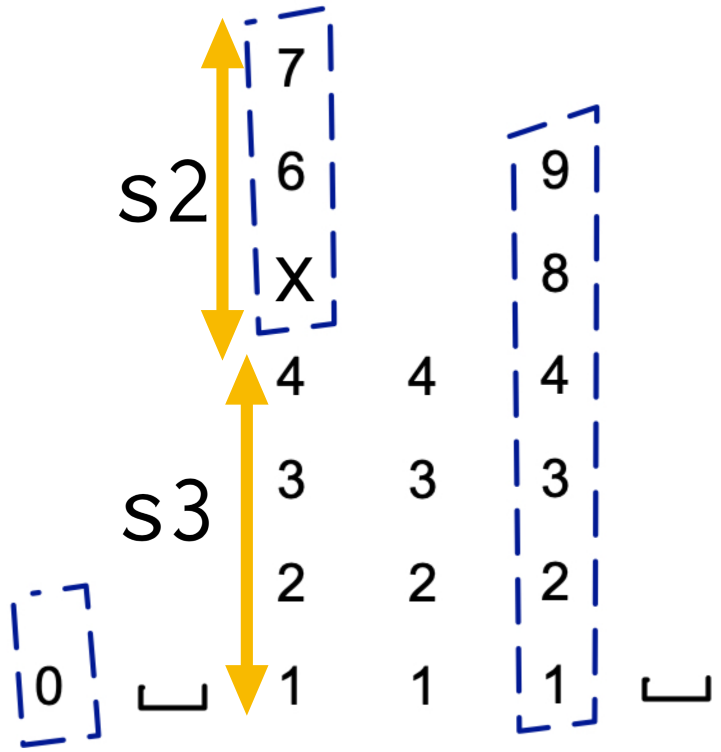
a vertex x is a **base** when $LOWLINK(x) \geq num[x]$

Program

```
type vertex
constant vertices: set vertex
function successors vertex : set vertex
type env = {stack: list vertex;
            sccs: set (set vertex);
            sn: int; num: map vertex int}
```

```
let tarjan () =
  let e = {stack = Nil; sccs = empty;
           sn = 0; num = const (-1)} in
  let (_, e') = dfs vertices e in e'.sccs
```

Program




```
let rec dfs1 x e =
  let n0 = e.sn in
  let (n1, e1) = dfs (successors x)
        (add_stack_incr x e) in
  if n1 < n0 then (n1, e1) else
    let (s2, s3) = split x e1.stack in
    (+∞, {stack = s3;
       sccs = add (elements s2) e1.sccs;
       sn = e1.sn; num = set_infty s2 e1.num})
```

```
with dfs r e = if is_empty r then (+∞, e) else
  let x = choose r in
  let r' = remove x r in
  let (n1, e1) = if e.num[x] ≠ -1
     then (e.num[x], e) else dfs1 x e in
  let (n2, e2) = dfs r' e1 in (min n1 n2, e2)
```

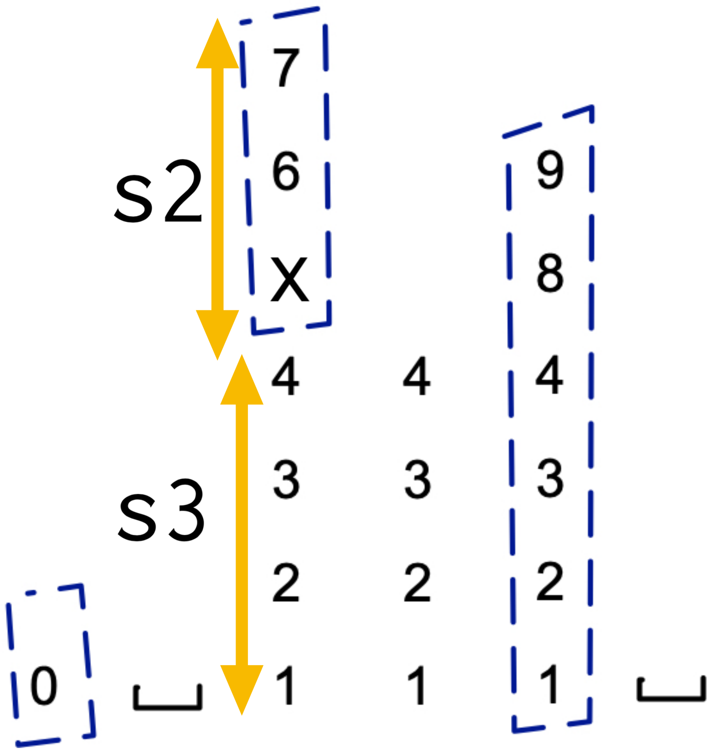
Proof

Program

```
type vertex
constant vertices: set vertex
function successors vertex : set vertex
type env = {ghost black: set vertex;
    ghost gray: set vertex;
    stack: list vertex; sccs: set (set vertex);
    sn: int; num: map vertex int}
```

```
let tarjan () =
  let e = {black = empty; gray = empty;
    stack = Nil; sccs = empty; sn = 0;
    num = const (-1)} in
  let (_, e') = dfs vertices e in e'.sccs
```

Program




```
let rec dfs1 x e =
  let n0 = e.sn in
  let (n1, e1) = dfs (successors x)
          (add_stack_incr x e) in
  if n1 < n0 then (n1, add_black x e1) else
    let (s2, s3) = split x e1.stack in
    (+∞, {stack = s3;
        black = add x e1.black; gray = e.gray;
        sccs = add (elements s2) e1.sccs;
        sn = e1.sn; num = set_infty s2 e1.num})
```

```
let add_stack_incr x e =
  let n = e.sn in
  {black = e.black; gray = add x e.gray;
   stack = Cons x e.stack; sccs = e.sccs;
   sn = n+1; num = e.num[x ←n]}
```

```
let add_black x e =
 {black = add x e.black; gray = remove x e.gray;
  stack = e.stack; sccs = e.sccs;
  sn = e.sn; num = e.num}
```

Invariant

(1) consistent colors

(2) consistent numbering

(3) vertices pairwise **distinct** in stack

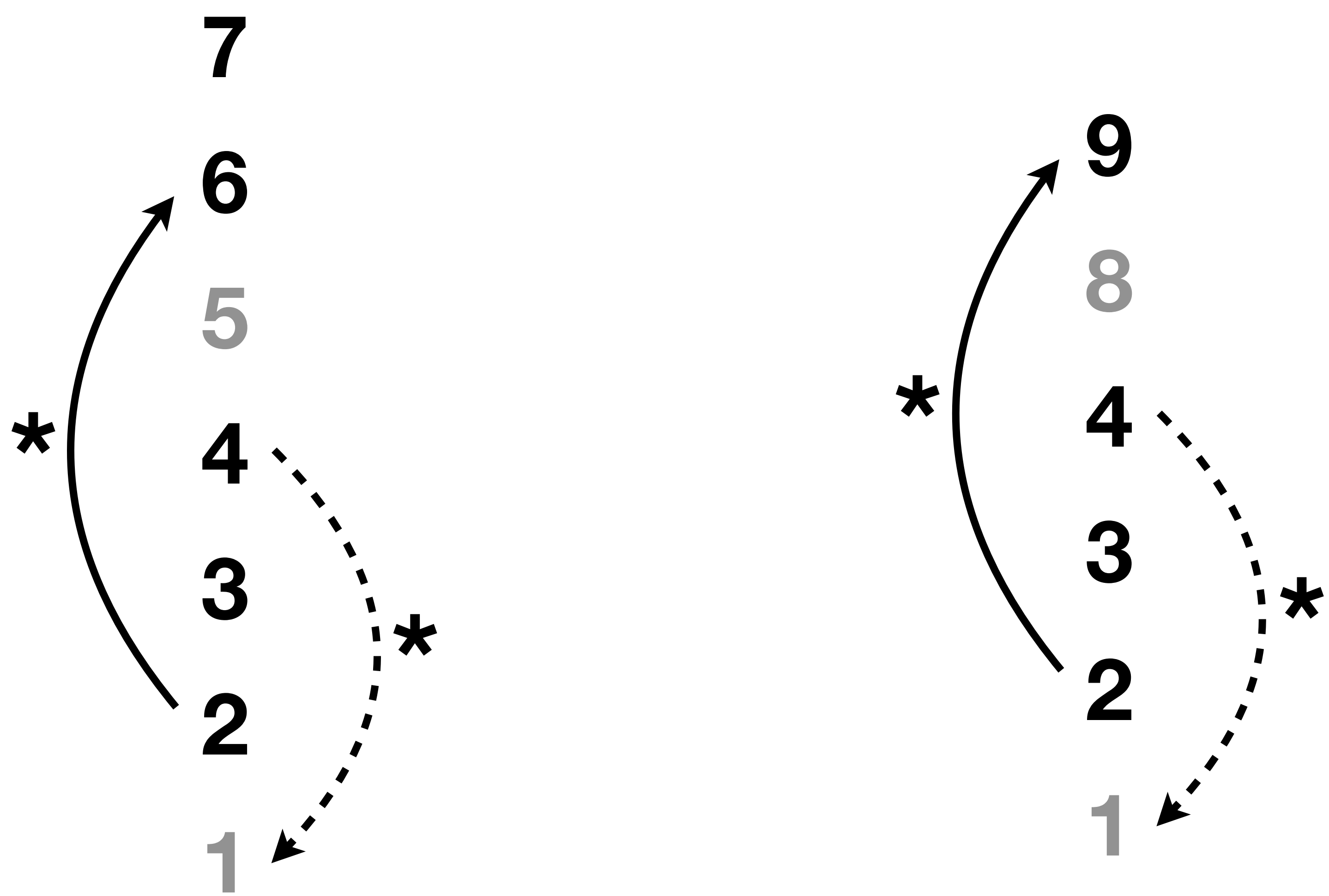
(4) no edge from black to white

(5) in stack any vertex reaches any **higher** vertex

(6) in stack any vertex reaches a **gray lower** vertex

(7) the sccs field is the set of **black** SCCs

Invariant



stack

Why3 Proof

Pre/Post-conditions

```
let rec dfs1 x e  =
(* pre-condition *)
requires{mem x vertices}
requires{∀y. mem y e.gray → reachable y x}
requires{not mem x (union e.black e.gray)}
requires{wf_env e} (* I *)



(* post-condition *)
returns{(_, e') → wf_env e' ∧ subenv e e'}
returns{(_, e') → mem x e'.black}
returns{(n, e') → n ≤ e'.num[x]}
returns{(n, e') → n = +∞ ∨ num_of_reachable n x e'}
returns{(n, e') → ∀y. xedge_to e'.stack e.stack y
                        → n ≤ e'.num[y]}
```

} *LOWLINK*

Assertions

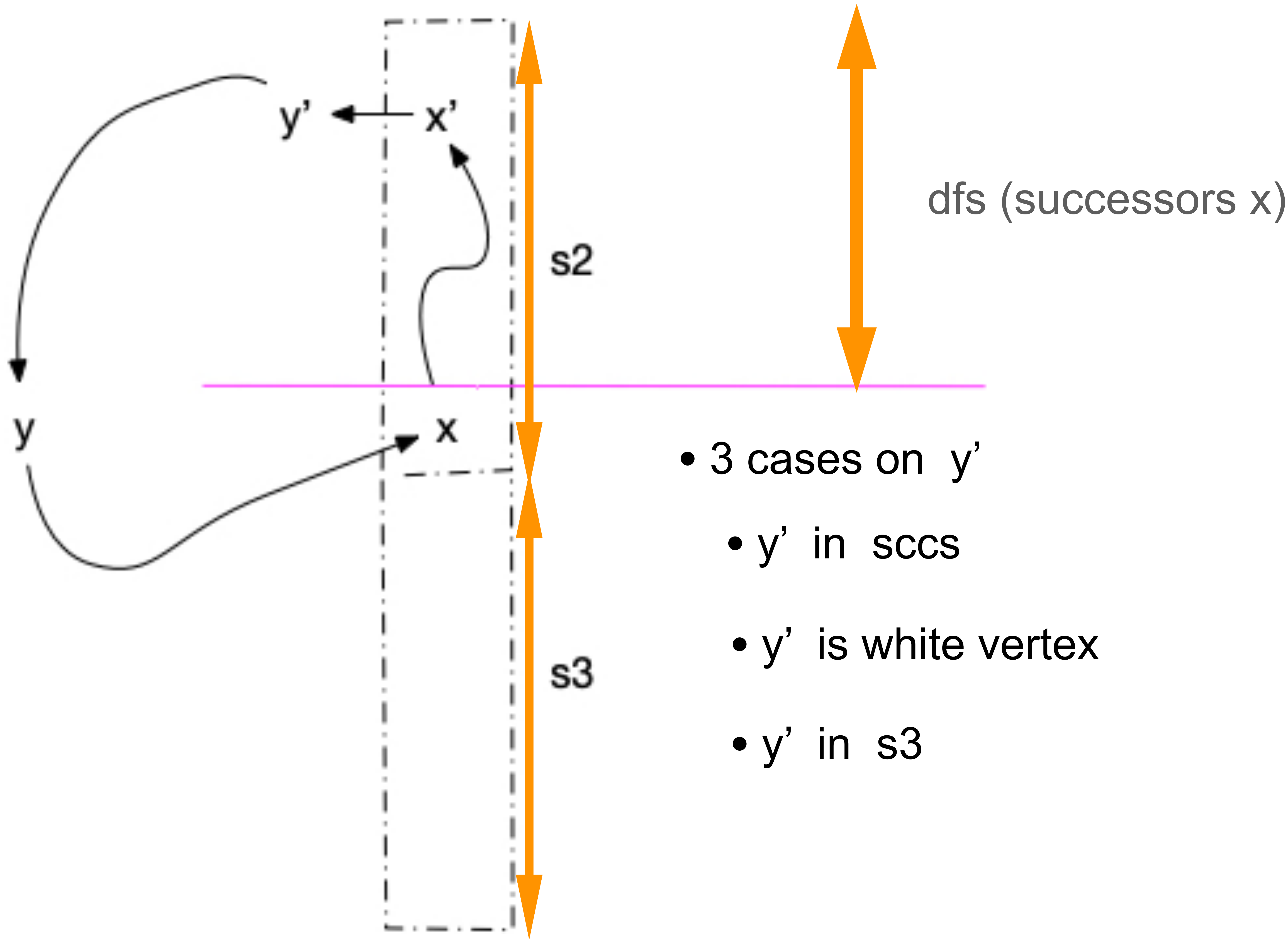
```
let n0 = e.sn in
let (n1, e1) = dfs (successors x) (add_stack_incr x e) in
if n1 < n0 then begin
  assert{n1 ≠ +∞};
  assert{∃y. y ≠ x ∧ mem y e1.gray ∧
          e1.num[y] < e1.num[x] ∧ in_same_scc x y};
  (n1, add_black x e1) end
```

Assertions

```
else
  let (s2, s3) = split x e1.stack in
  assert{is_last x s2 ∧ s3 = e.stack ∧
     subset (elements s2) (add x e1.black)};
  assert{is_subscc (elements s2)};
  assert{∀y. in_same_scc y x → lmem y s2};
  assert{is_scc (elements s2)};
  assert{inter e.gray (elements s2) == empty};
  (+∞, {black = add x e1.black; gray = e.gray;
     stack = s3; sccs = add (elements s2) e1.sccs;
     sn = e1.sn; num = set_infty s2 e1.num})
```

completeness

Completeness proved in Coq



- 3 cases on y'
  - y' in sccs
  - y' is white vertex
  - y' in s3

Assertions

provers	Alt-Ergo	CVC4	E-prover	Z3	#VC	#PO
49 lemmas	1.91	26.11	3.33		70	49
split	0.09	0.16			6	6
add_stack_incr	0.01				1	1
add_black	0.02				1	1
set_infty	0.03				1	1
dfs1	77.89	150.2	19.99	13.67	79	20
dfs	4.71	3.52		0.26	58	25
tarjan	0.85				15	5
total	85.51	179.99	23.32	13.93	231	108

Table 1. Performance results of the provers (in seconds, on a 3.3 GHz Intel Core i5 processor). Total time is 341.15 seconds. The two last columns contain the numbers of verification conditions and proof obligations. Notice that there may be several VCs per proof obligation.

+ 2 Coq proofs (16 loc + 141 loc)

Coq Proof

Functions

```
Record env := Env {black : {set V};
                   stack : seq V;
                   sccs : {set {set V}};
                   sn : nat;  num : {ffun V ➔ nat}}.
```

```
Definition dfs1  dfs  x  e :=
 let: (n1, e1) :=
  dfs [set y  in  successors  x] (add_stack  x  e) in
  if  n1 < sn e  then  (n1,  add_black  x  e1)
  else (infty,  add_sccs  x  e1).
```

```
Definition dfs  dfs1  dfs'  r  e :=
 if  [pick x in r]  isn't  Some x  then  (infty, e)
 else  let r' := r :\ x in
        let: (n1, e1) :=
          if num  e  x != 0 then (num  e  x, e)  else  dfs1  x  e in
        let: (n2, e2) := dfs'  r'  e1   in   (minn  n1  n2,  e2).
```

Functions

```
Fixpoint tarjan_rec n :=
 if n is n1.+1 then
  dfs (dfs1 (tarjan_rec n1)) (tarjan_rec n1)
 else fun r e => (infty, e).
```

```
Let N := #|V| * #|V|.+1 + #|V|.
Definition tarjan := sccs (tarjan_rec N setT e0).2.
```

Proof

```
Definition dfs_correct
    (dfs : {set V} → env → nat * env)  r  e  :=
    pre_dfs  r  e  →
    let (n, e') := dfs  r  e  in  post_dfs  r  e  e'  n.
```

```
Definition dfs1_correct
    (dfs1 : V → env → nat * env)  x  e :=
    (x ∈ white e)  →  pre_dfs  [set x]  e →
    let (n, e') :=  dfs1  x  e  in  post_dfs  [set  x]  e  e'  n.
```

Proof

```
Lemma dfs_is_correct dfs1' dfs' (r : {set V}) e :
  (∀x, x ∈ r → dfs1_correct dfs1' x e) →
  (∀x, x ∈ r → ∀e1, white e1 \subset white e →
        dfs_correct dfs' (r :\ x) e1) →
  dfs_correct (dfs dfs1' dfs') r e.
```

```
Lemma dfs1_is_correct dfs' (x : V) e :
  (dfs_correct dfs' [set y | edge x y] (add_stack x e)) →
  dfs1_correct (dfs1 dfs') x e.
```

```
Theorem tarjan_rec_terminates n r e :
n ≥ #|white e| * #|V|.+1 + #|r| →
dfs_correct (tarjan_rec n) r e.
```

Another Coq proof

- Coq with Ssreflect + Mathematical Components

```
Definition gsymconnect x y := gconnect x y && gconnect y x.
Definition gsccs := equivalence_partition gsymconnect [set: V].
```

Another Coq proof

```
Record env := Env {esccs : {set {set V}}; num: {ffun V → nat}}.

Definition dfs1 dfs x e :=
   let: (n1, e1) as res := dfs (successors x) (visit x e) in
   if n1 < sn e then res else (∞, store (stack e1 \ stack e) e1).
Definition dfs dfs1 dfs (roots : {set V}) e :=
  if [pick x in roots] isn't Some x then (∞, e) else
  let: (n1, e1) := if num e x ≤ ∞ then (num e x, e) else dfs1 x e in
  let: (n2, e2) := dfs (roots \ [set x]) e1 in (minn n1 n2, e2).
```

$0 \le num[x] < \infty$	$num[x] = \infty$	$num[x] = \infty + 1$
stack e	*sccs*	white

Another Coq proof

```
Definition wf_env e := [∧ esccs e ⊆ gsccs,
  ∀x, num e x < ∞ → num e x < sn e,
  ∀x, (num e x = ∞) = (x ∈ cover (esccs e)) &
  ∀x y, num e x ≤ num e y < sn e → gconnect x y].
```

```
Definition subenv e1 e2 := [∧ esccs e1 ⊆ esccs e2,
  ∀x, num e1 x < ∞ → num e2 x = num e1 x &
  ∀x, num e2 x < sn e1 → num e1 x < sn e1].
```

```
Definition outenv (roots : {set V}) (e e' : env) := [∧
  ∀x y, x ∈ stack e' \ stack e → y ∈ stack e' \ stack e → gconnect x y,
  ∀x, x ∈ stack e' \ stack e → ∃y, y ∈ stack e ∧ gconnect x y &
  visited e' = visited e ∪ nexts (∼: visited e) roots ].
```

Isabelle/HOL Proof

Proof

```
function (domintros) dfs1  and  dfs  where
 dfs1 x e =
  (let (n1, e1) = dfs (successors x) (add_stack_incr x e) in
   if  n1 < int (sn e) then  (n1, add_black x e1)
   else (let (l, r) = split_list x (stack e1)  in
     (+∞, (| black = insert x (black e1), gray = gray e,
             stack = r, sn = sn e1, sccs = insert (set l) (sccs e1),
             num = set_infty l (num e1) |) )))

and
dfs roots e =
  (if roots = {} then (+∞, e)
   else (let x = SOME x . x ∈ roots;
         res1 = (if num e x ≠ -1 then (num e x, e) else dfs1 x e);
         res2 = dfs (roots - {x}) (snd res1)
    in   (min (fst res1) (fst res2),  snd res2) ))
```

Proof

definition colored_num **where** colored_num e ≡

$\forall v \in$ colored e. $v \in$ vertices $\wedge$ num e v $\neq -1$

theorem dfs1_dfs_termination :

[x ∈ vertices - colored e; colored_num e] ⟹ dfs1_dfs_dom (Inl(x, e))

[r ⊆ vertices; colored_num e] ⟹ dfs1_dfs_dom (Inr(r, e))

theorem dfs_partial_correct:

[dfs1_dfs_dom (Inl(x, e)); dfs1_pre x e] ⟹ dfs1_post x e (dfs1 x e)

[dfs1_dfs_dom (Inr(r, e)); dfs_pre r e] ⟹ dfs_post r e (dfs r e)

theorem dfs_correct:

dfs1_pre x e ⟹ dfs1_post x e (dfs1 x e)

dfs_pre r e ⟹ dfs_post roots e (dfs r e)

Conclusion

Why3 - Coq - Isabelle

	why3	coq	isabelle/HOL
expressivity	-	++	+
readability	+++	-	+
stability	-	+++	+
ease of use	-	-	-
automation	++	-	+
partial correctness	+++	- -	-
code extraction	++	+	-
trusted base	-	+++	+++
# lines auto	392	0	? (314ui)
# lines manual	157	1535	1690

… other systems ?

`http://www-sop.inria.fr/marelle/Tarjan/contributions.html`

Todo list

- proof of implementation
- other algorithms (biconnected, planarity, minimum spanning tree)
- teaching