The cost of usage in the λ-calculus

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Plan

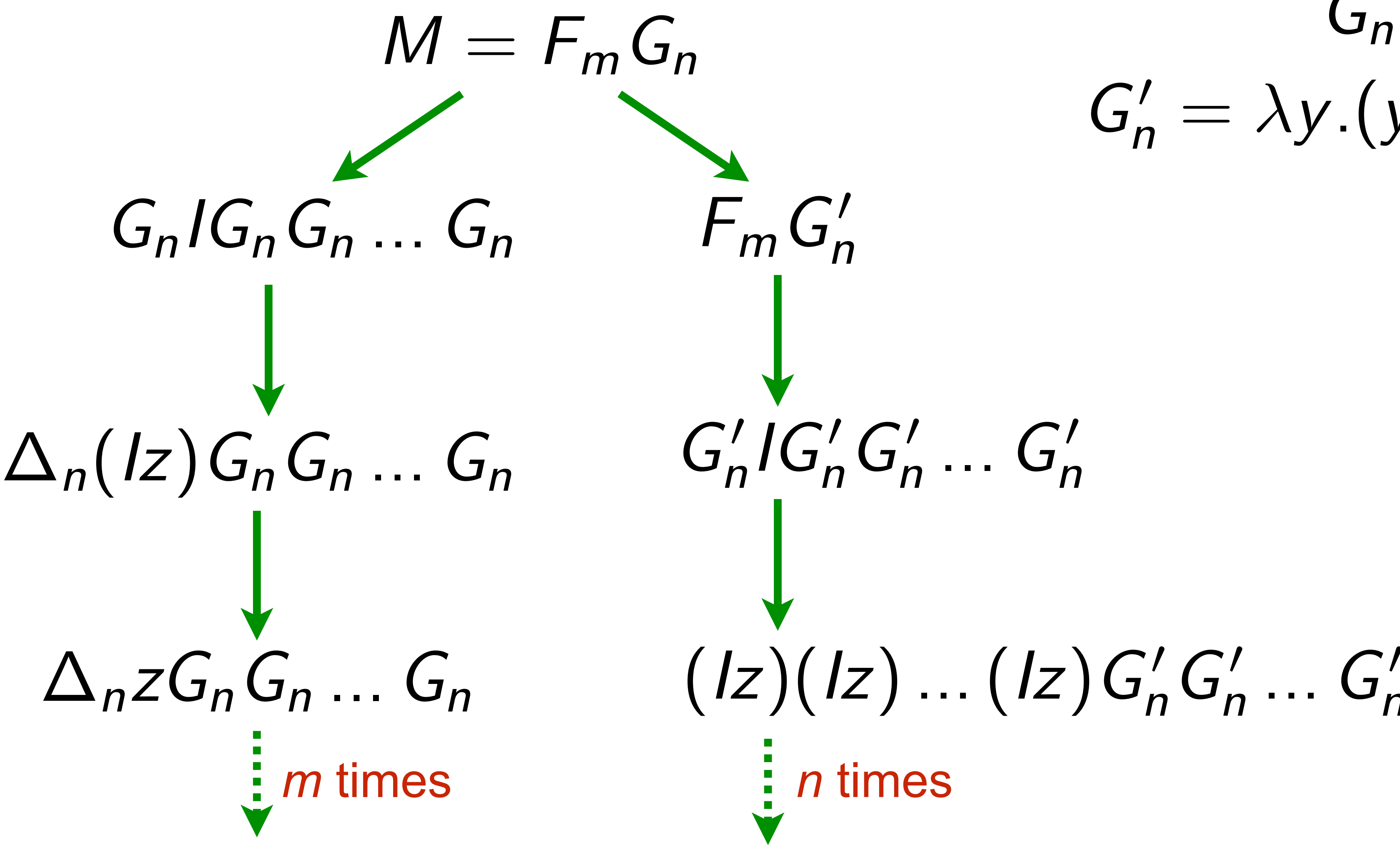
- the standardization theorem (with upper bounds)
- our result
- rigid and minimum prefixes (stability thm)
- Xi's proof (with upper bounds)
- Xi's proof revisited with live occurences

.. joint work with Andrea Asperti (LICS 2013) ..

Shortest reductions

$$F_m = \lambda x.xIxx\ldots x$$
$$\Delta_n = \lambda x.x\,x\ldots x$$
$$G_n = \lambda y.\Delta_n(yz)$$
$$G'_n = \lambda y.(yz)(yz)\ldots(yz)$$

- non effective strategies



Standardization

Standard reductions (1/4)

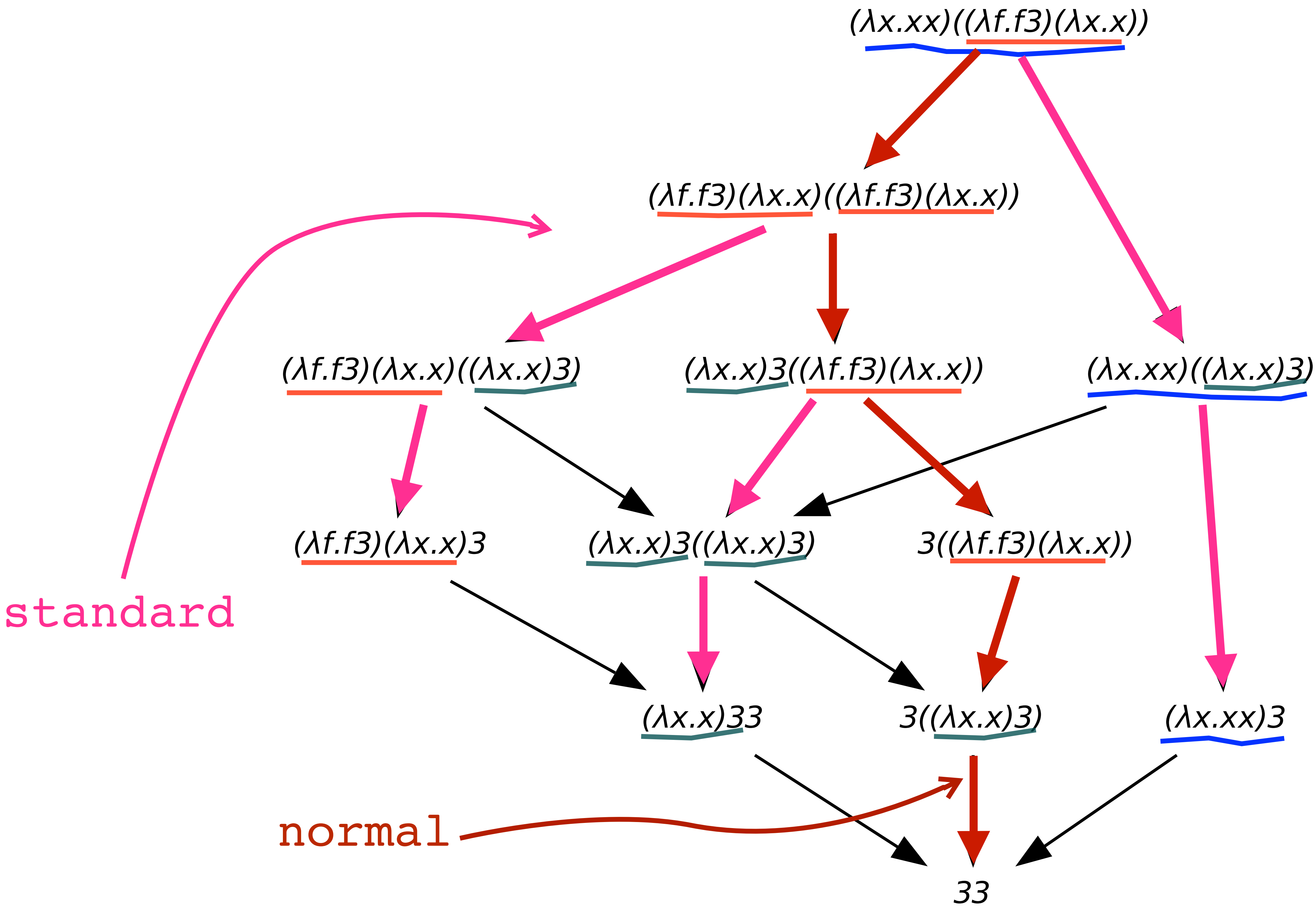
- **Definition:** The following reduction is **standard**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R_j' to the left of R_i in M_{i-1}.

- **Definition:** The leftmost-outermost reduction is also called the **normal reduction.**

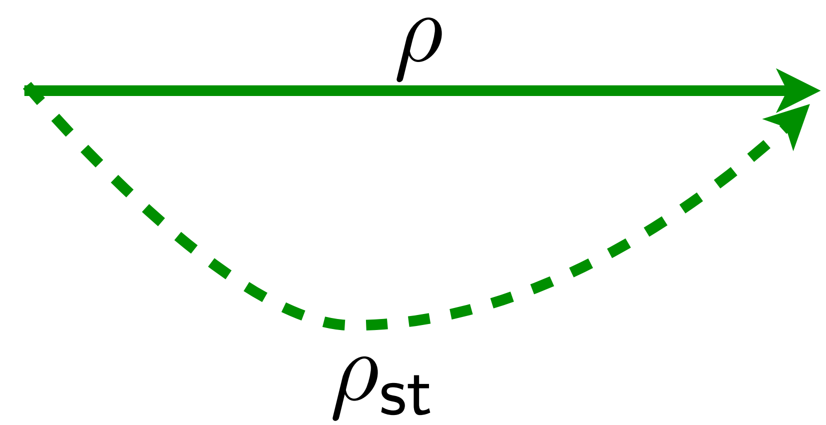
Standard reductions (2/4)



Standard reductions (3/4)

- **Standardization thm** [Curry 50]

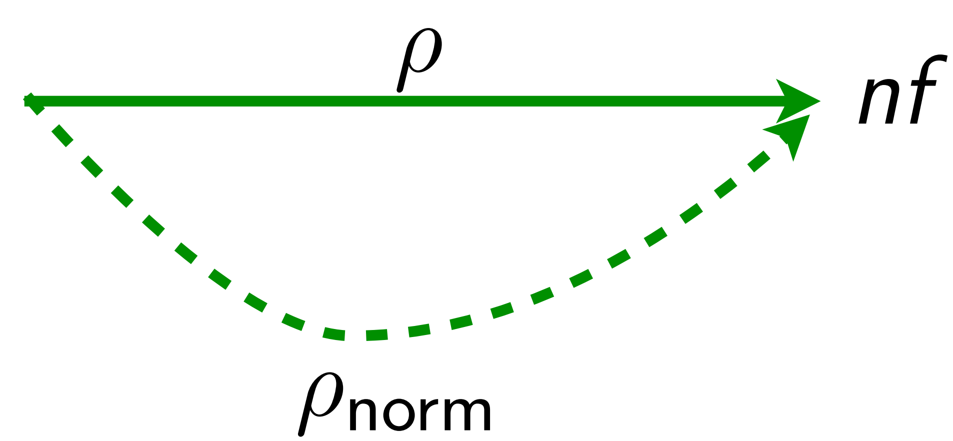
Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\text{st}]{\star} N$.



Any reduction can be performed outside-in and left-to-right.

- **Normalization corollary**

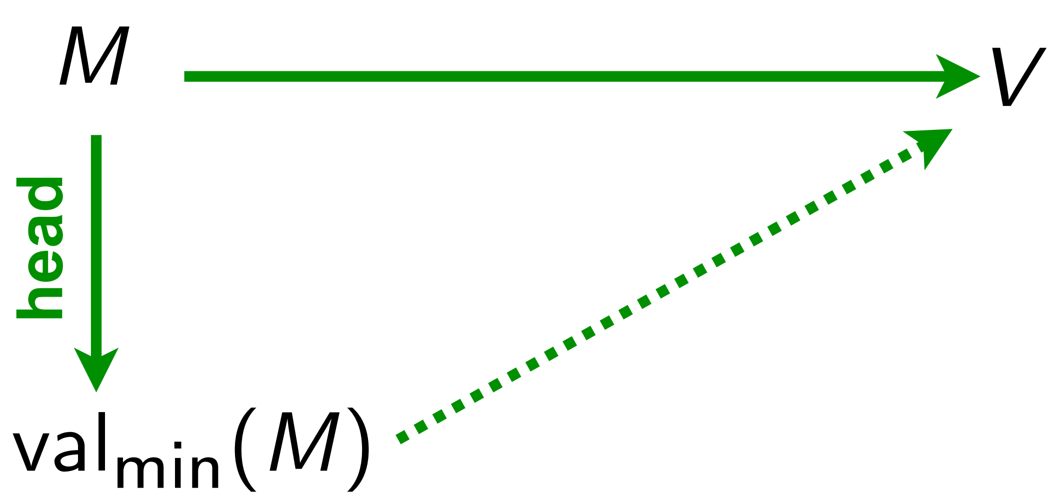
Let $M \xrightarrow{\star} nf$. Then $M \xrightarrow[\text{norm}]{\star} nf$.



Standard reductions (4/4)

- **Head reduction corollary for values**

Let $M \xrightarrow{*} V$. Then $M \xrightarrow[\text{head}]{*} \mathrm{val}_{\mathrm{min}}(M) \xrightarrow{*} V$



Our result

- **Upper-bound on standard reductions** [Hongwey Xi, 99]

  Let $\ell = |\rho|$ and $\rho : M \xrightarrow{\star} N$. Then $|\rho_{st}| \leq |M|^{2^{\ell}}$

  where $\rho_{st} : M \xrightarrow[\text{st}]{\star} N$.

- **Upper-bound to normal forms** [Asperti-JJL, 13]

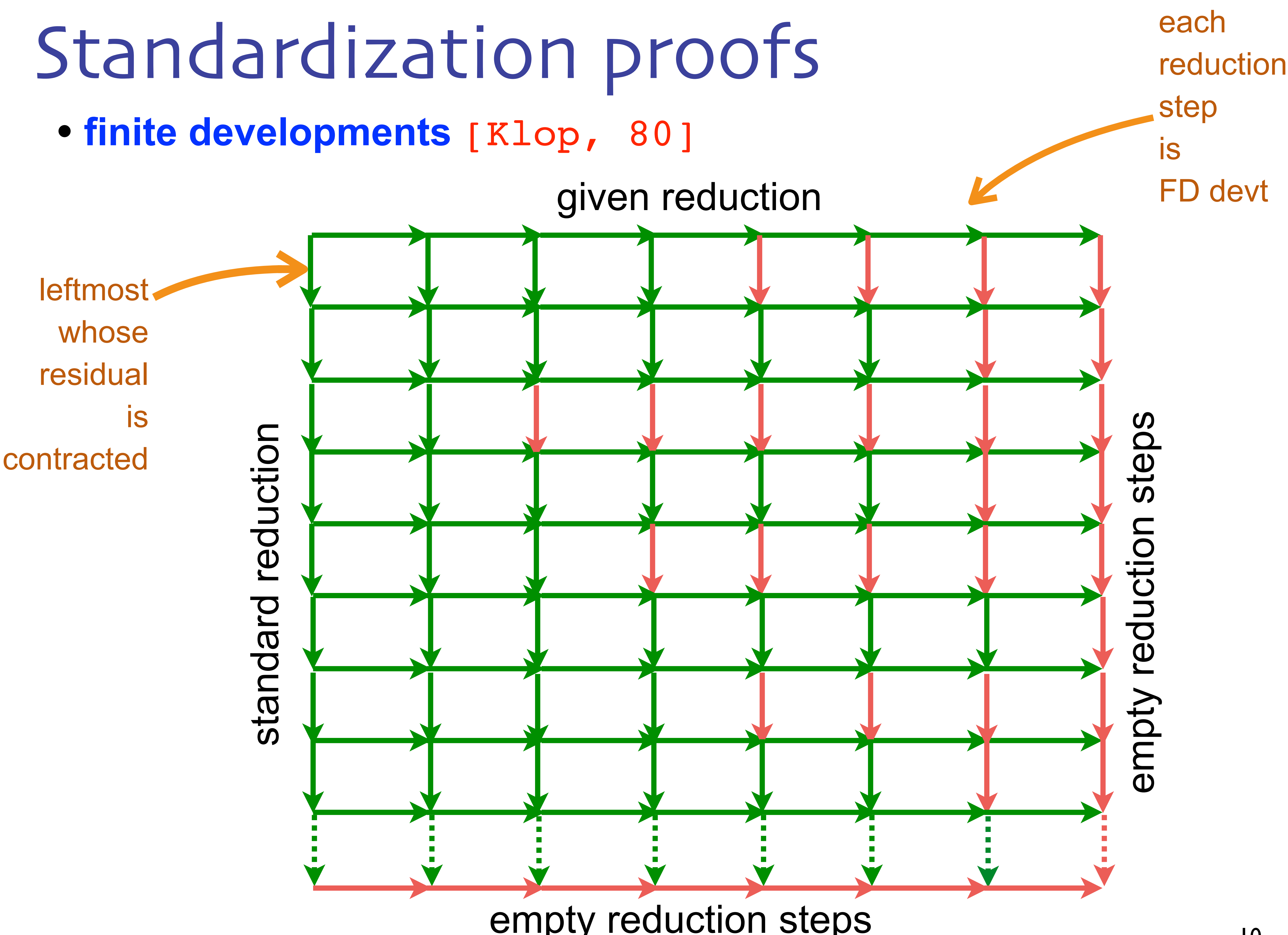
  Let $\ell = |\rho|$ and $\rho : M \xrightarrow{\star} x$. Then $|\rho_{norm}| \leq \ell!$

  where $\rho_{norm} : M \xrightarrow[\text{norm}]{\star} x$.

We gain one exponential.

Standardization proofs

- **finite developments** [Klop, 80]



Standardization proofs

- **finite developments** [Gonthier-Melliès-JJL,92]

  tricky axiomatic proof

- **head normal forms** [Mitschke,80]

- **initial proof and statement** [Curry&Feys,70]

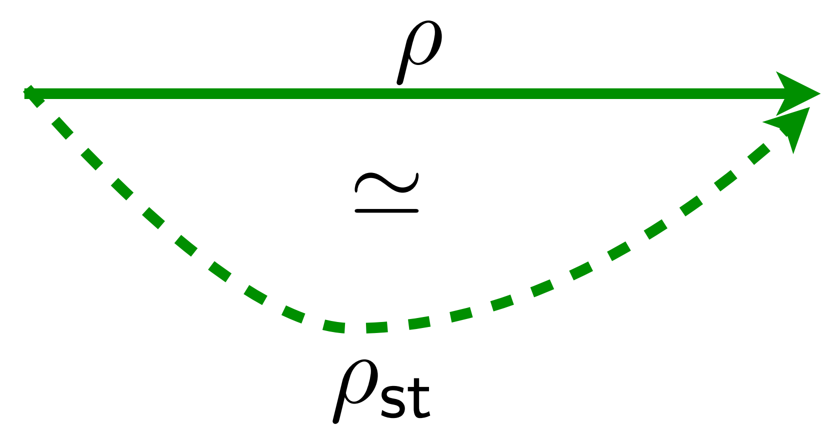
  correct statement, but proof ?

Standard reductions (4+/4)

- **Standardization thm** [JJL 77]

  Let $\rho : M \xrightarrow{\star} N$. $\exists! \rho_{st}.\ M \xrightarrow[\text{st}]{\star} N$

  and $\rho_{st} \simeq \rho$.



Standard reduction is canonical representative in permutation class.

- **λI-standardization** [Church 36]

  Standard reduction is longest in its equivalence class.

Rigid prefixes:

stability

and

multiplicity of variables

Stability (1/2)

- **Definition** `[rigid prefix]` Any rigid prefix A of M is any prefix of M where never the left of an application can reduce to an abstraction.

$$M = \Omega(\lambda x.x(Ix))(IIx)$$

$$\Omega = (\lambda x.xx)(\lambda x.xx)$$

$$I = \lambda x.x$$

$\_(\lambda x.x\_)\_$ rigid prefix of M

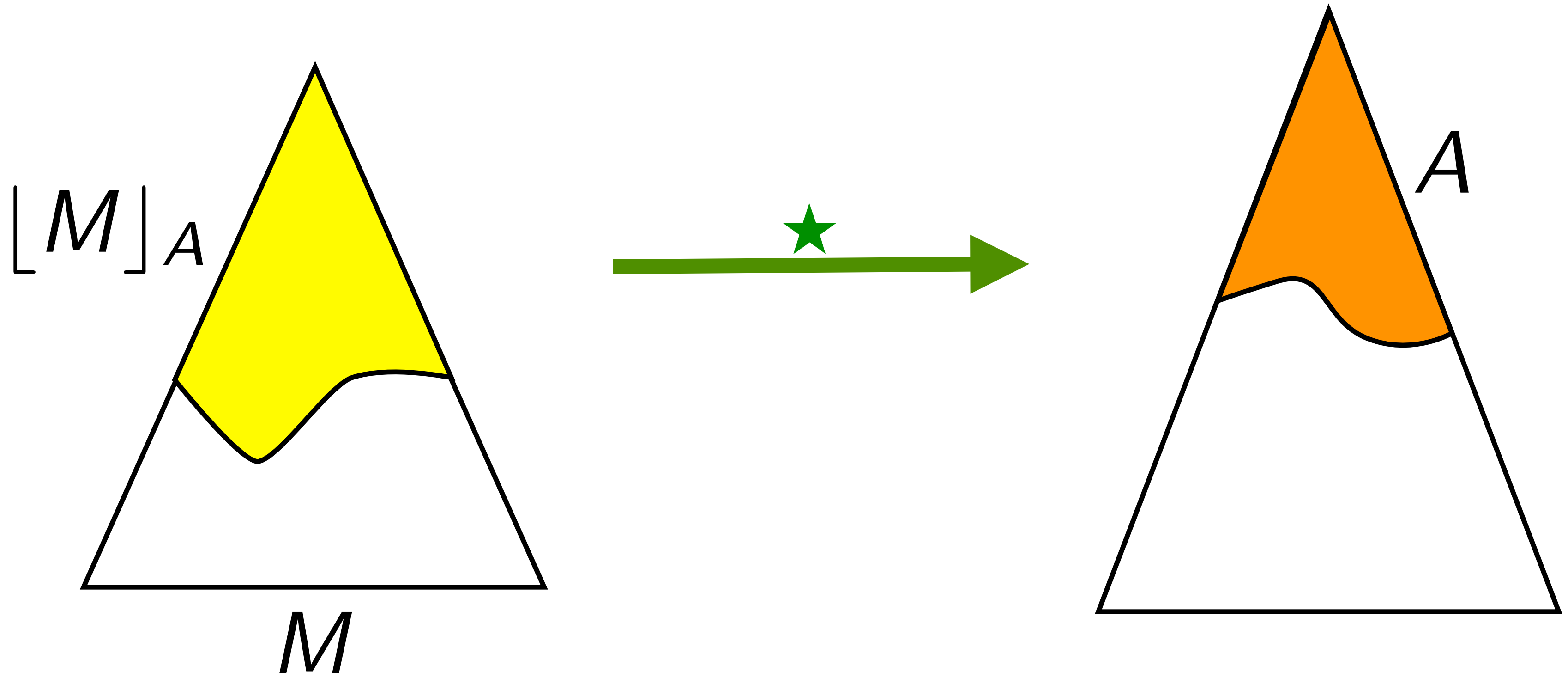
$\_(\lambda x.x\_)(\_Ix)$ not rigid prefix of M

(rigid prefixes are finite prefixes of Berarducci trees)

- **Definition** M produces A if $M \xrightarrow{\star} N$ and A is rigid prefix of N.

Stability (2/2)

- **Theorem** `[stability]` For any rigid prefix A produced by M, there is a unique minimal prefix $\lfloor M \rfloor_A$ of M producing A.



- **Fact** `[monotony]` Let M produce A rigid and $M \xrightarrow{\star} N$. Then N produces A.

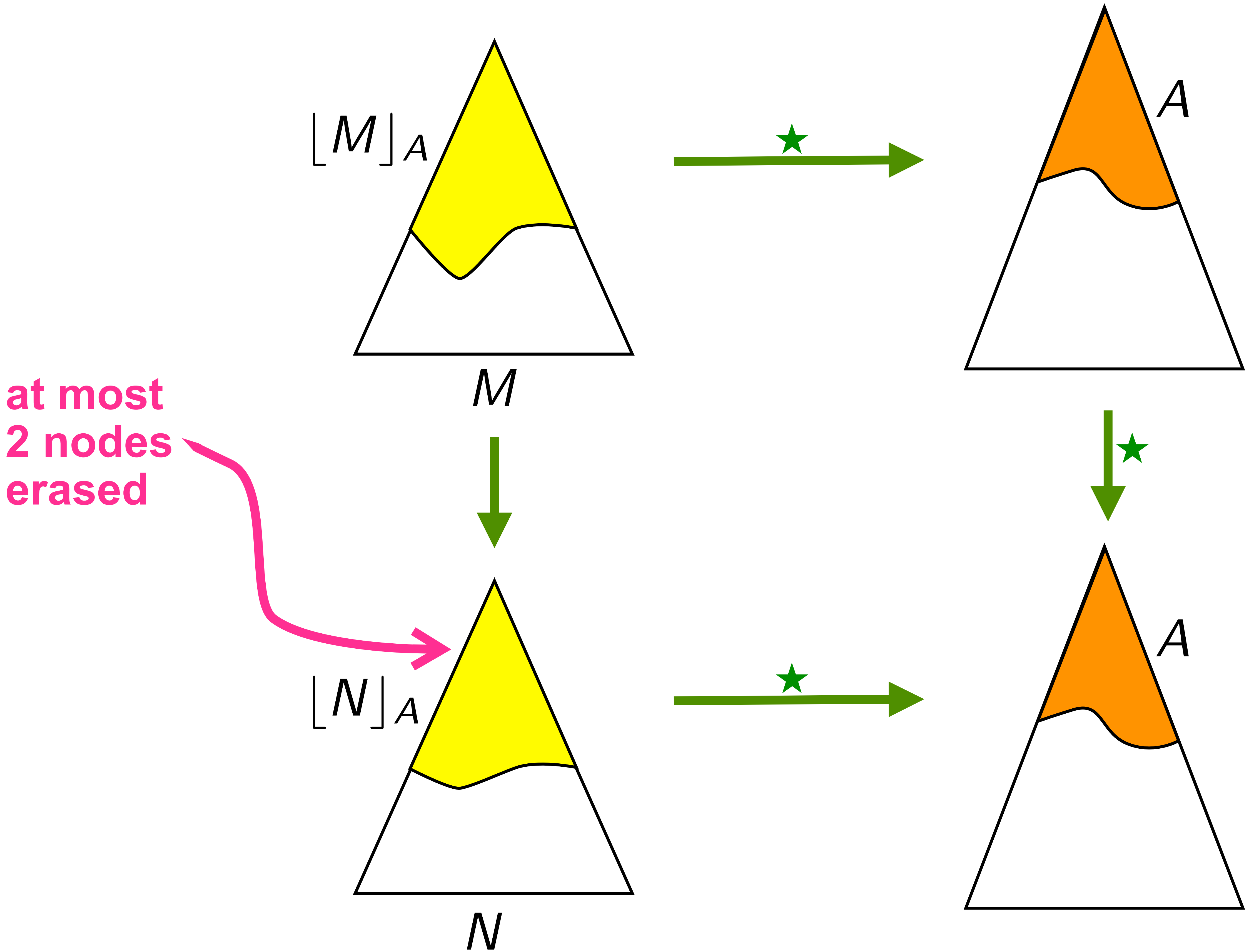
Slow consumption (1/2)

- **Lemma 1** `[slow consumption]` Let M produce A rigid and $M \longrightarrow N$. Then $|\lfloor N \rfloor_A| \geq |\lfloor M \rfloor_A| - 2$.

  i.e. $|\lfloor M \rfloor_A|_@ \leq 1 + |\lfloor N \rfloor_A|_@$ where $|P|_@$ is the applicative size of P (its number of application nodes).

- **Corollary** Let $\rho : M \stackrel{*}{\longrightarrow} N$ and A be rigid prefix of N. Then $|\lfloor M \rfloor_A|_@ \leq |\rho| + |A|_@$.

Slow consumption (2/2)



Multiplicity of variables

- **Definition** Let M produce A rigid. An occurrence of x is live for A if it belongs to $\lfloor M \rfloor_A$.

Let $m_A(x)$ be the number of live occurrences of x in M.

We pose $m_A(R) = m_A(x)$ when $R = (\lambda x.M)N$.

- **Lemma 2** [upper bound on live multiplicity]

Let $\rho : M \xrightarrow{\star} N$ and A rigid prefix of N. Then

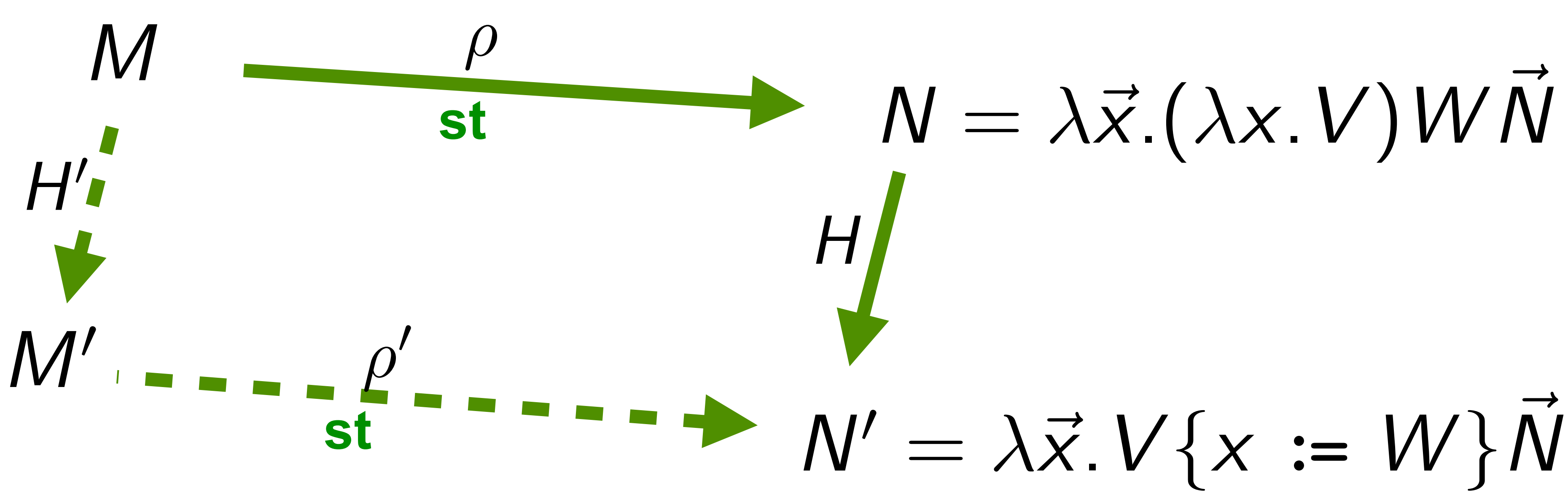
$m_A(x) \leq |\rho| + |A|_@ + 1$ for any variable x in M.

Standardization

Xi's proof of standardization (1/2)

- **Lemma** `[reordering of head redexes]` H is residual of H'.

  Then



  with $|\rho'| \leq \lceil 1, m(H) \rceil . |\rho|$

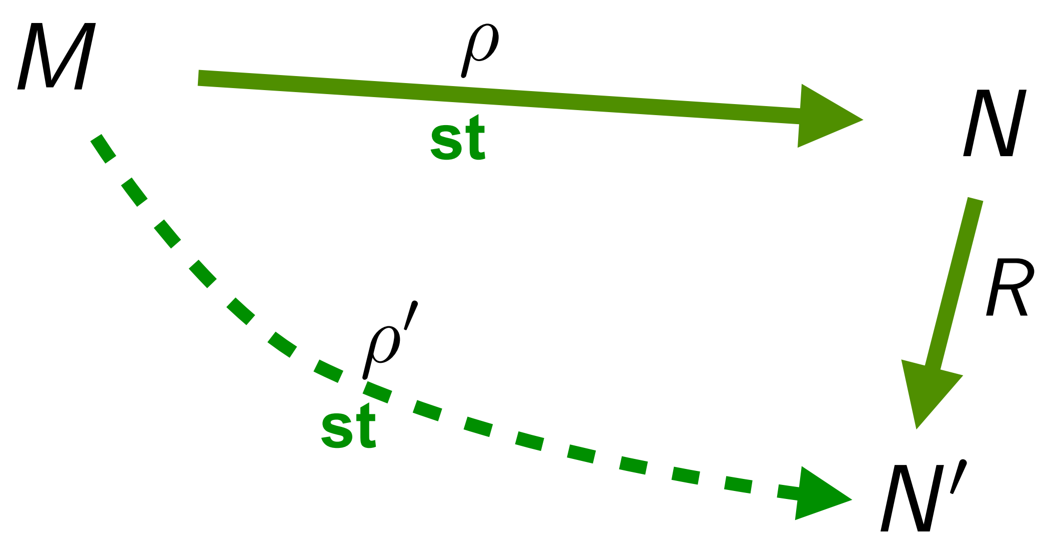
Proof Easy since $M = \lambda\vec{x}.(\lambda x.T)U\vec{M}$ and $\rho = \rho_T \rho_U \rho_1 \cdots \rho_n$.

And ρ' is disjoint intermix of ρ_T, several ρ_U, followed by ρ_i's.

Thus $|\rho'| = |\rho_T| + m(H).|\rho_U| + \sum_i |\rho_i|$

Xi's proof of standardization (2/2)

- **Corollary**



with $|\rho'| \leq 1 + \lceil 1, m(R) \rceil . |\rho|$

Proof
By induction on pair $(|\rho|, |M|)$. Cases on ρR contracting head redex or not + previous lemma.

Xi's proof of standardization (2/2)

- **Theorem** [standardization with upper bounds]

  Let $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$

  Then there is ρ standard from M to N such that

  $|\rho| \leq (1 + \lceil 1, m(R_2) \rceil)(1 + \lceil 1, m(R_3) \rceil) \cdots (1 + \lceil 1, m(R_n) \rceil)$

  Proof By induction on the length n of reduction from M to N.

Proof of our upper bound (1/2)

- **Theorem** `[standardization with upper bounds]`
  Let $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$
  and A be rigid prefix of N.
  Then there is ρ standard from M to N' such that
  $|\rho| \leq (1 + \lceil 1, m_A(R_2) \rceil)(1 + \lceil 1, m_A(R_3) \rceil) \cdots (1 + \lceil 1, m_A(R_n) \rceil)$
  and A is rigid prefix of N'.

Proof of our upper bound (2/2)

- **Corollary 1** Let $\rho : M \xrightarrow{\star} N$ and A be rigid prefix of N. Then there is ρ_{st} standard such that:

$$|\rho_{st}| \leq \frac{(|\rho| + |A|_{@})!}{(1 + |A|_{@})!}$$

Proof Simple calculation with lemma 2 and previous thm.

- **Corollary 2** Let $\rho_{st} : M \xrightarrow{\star} x$ be standard reduction. Then $|\rho_{st}| \leq |\rho|!$ where ρ is shortest reduction from M to x.

Conclusions

Conclusion

- terms are easy to grow in the λ-calculus
- but take time to consume terms

→ there is a need for sharing

- back to earth …. and higher-order functional languages