Small families

(at INRIA with Gérard and in the historical λ-calculus)

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caml

years

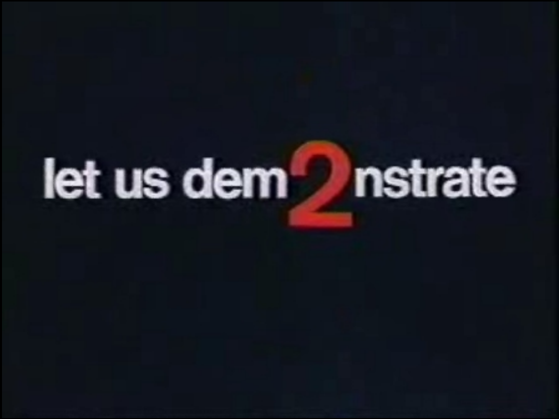
coq

sixty years is
31,557,600 minutes
and **one minute** is a
long time . . . .


let us dem0nstrate



let us dem1nstrate


let us dem2nstrate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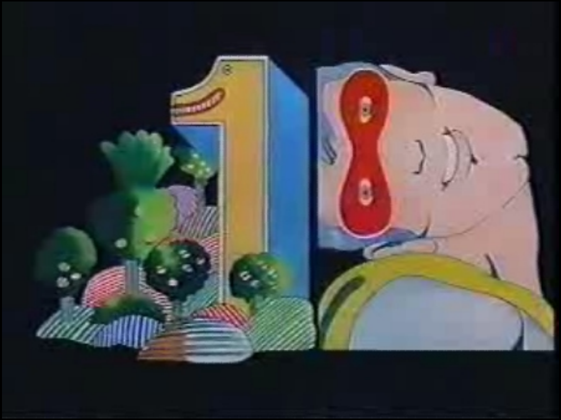
he passes the baccalauréat.

he writes his first program

and discovers functional programming.

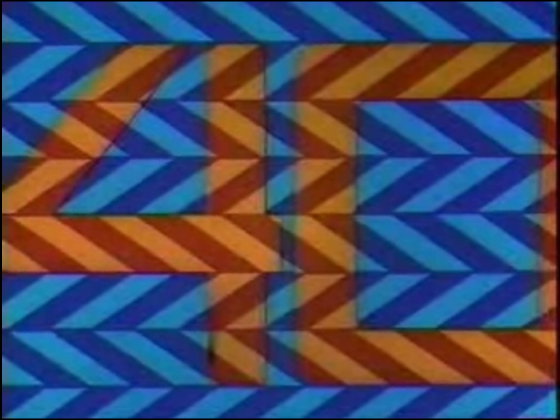


he now is an attractive researcher

and starts a glorious academic life.


32


संस्कृत भारती
सं



he passes 40 without care · · ·

· · · **still some hope** · · ·

· · · **getting anxious** · · ·

· · · **and done !**

Plan

1. strong normalisation
2. finite developments
3. redex families
4. generalised finite developments
5. conclusion

Strong normalisation

Typed λ-calculus

Theorem (Church-Rosser)

The typed λ-calculus is confluent.

and

Theorem (strong normalization)

In typed λ-calculus, there are no infinite reductions.

True at 1st order (Curry/Church), 2nd order (system F), $\cdots$ 60th order and even more (F_ω, Coq).

Corollary

The typed λ-calculus is a canonical system.

The classical λ-calculus is confluent but provides infinite reductions : let $\Delta = \lambda x.xx$, then $\Omega = \Delta\Delta \to \Delta\Delta = \Omega$.

Hyland-Wadsworth's D_∞-like λ-calculus

The idea is that $f_{n+1}(x) = (f(x_n))_n$ for $f, x \in D_\infty$

For the λ-calculus :

labels	$m, n, p \geq 0$
expressions	$M, N := x^n \mid (MN)^n \mid (\lambda x.M)^n$
β-conversion	$((\lambda x.M)^{n+1} N)^p \longrightarrow M\{x := N_{[n]}\}_{[n][p]}$

projection	$x^m_{[n]} = x^p$	substitution	$x^n\{x := P\} = P_{[n]}$
	$(MN)^m_{[n]} = (MN)^p$		$(MN)^n\{x := P\} = (M\{x := P\}N\{x := P\})^n$
	$(\lambda x.M)^m_{[n]} = (\lambda x.M)^p$		$(\lambda y.M)^n\{x := P\} = (\lambda y.M\{x := P\})^n$
	where $p = \lfloor m, n \rfloor$		

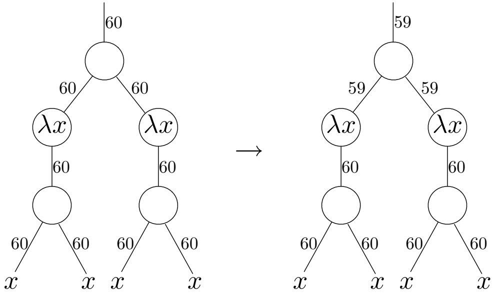
Notice that $((\lambda x.x^{58})^0 y^{59})^{60}$ is in normal form.

Degree of $((\lambda x.M)^n N)^p$ is n.

Hyland-Wadsworth's D_∞-like λ-calculus

Let $\Omega_n = (\Delta_n \Delta_n)^n$, $\Delta_n = (\lambda x.(x^{60} x^{60})^{60})^n$

$\Omega_{60} =$



$= \Omega_{59}$

Then

$$\Omega_{60} \to \Omega_{59} \to \Omega_{58} \to \cdots \Omega_1 \to \Omega_0 \text{ in normal form}$$

Up-to-60

β-conversion $\quad ((\lambda x.M)^{n} N)^p \rightarrow M\{x := N_{[n+1]}\}_{[n+1][p]}$
when $n \leq 60$

elevation

$$x^m_{[n]} = x^p$$

$$(MN)^m_{[n]} = (MN)^p$$

$$(\lambda x.M)^m_{[n]} = (\lambda x.M)^p$$

where $p = \lceil m, n \rceil$

substitution

$$x^n\{x := P\} = P_{[n]}$$

$$(MN)^n\{x := P\} = (M\{x := P\}N\{x := P\})^n$$

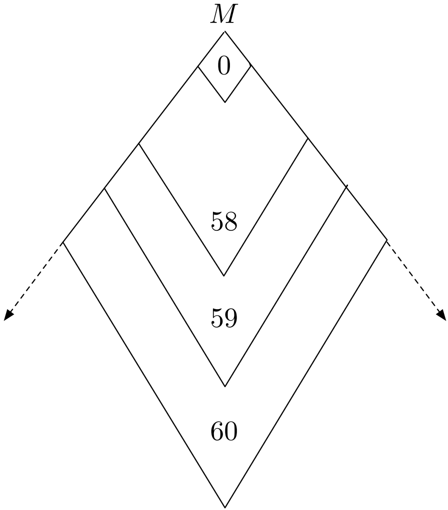
$$(\lambda y.M)^n\{x := P\} = (\lambda y.M\{x := P\})^n$$

Theorem (Church-Rosser+SN)

Hyland-Wadsworth and up-to-60 calculi are canonical systems.

Comes from associativity of min/max since $\lfloor m, \lfloor n, p \rfloor \rfloor = \lfloor \lfloor m, n \rfloor, p \rfloor$, and residuals keep degrees.

Compactness and canonical systems



Any reduction graph $\mathcal{R}(M)$ can be approximated by an increasing chain of reduction graphs $\mathcal{R}_0(M)$, $\mathcal{R}_1(M)$, ... $\mathcal{R}_{58}(M)$, $\mathcal{R}_{59}(M)$, $\mathcal{R}_{60}(M)$, ... of canonical systems.

Finite developments

Finite developments

Reductions of a set $\mathcal{F}$ of redexes in M are described by :

- putting 0 on degrees of redexes in $\mathcal{F}$,
- putting 60 on degrees of other redexes,
- applying the *up-to-1* calculus.

Theorem (finite developments — lemma of parallel moves)

There are no infinite reductions of a set $\mathcal{F}$ of redexes in M. All developments end on same term.

Proof : obvious since *up-to-1* is a canonical system.

Theorem (finite developments+ — the cube lemma)

The notion of residuals is consistent with finite developments.

Created redexes

- Let M_0 have all subterms labeled by 0,
  let $M_0 \twoheadrightarrow N$ and R redex in N of non-zero degree,
  then R is new redex (or *created* redex)

- Let $M = (\lambda x.x)(\lambda x.x)y$

  Then $M_0 = (((\lambda x.x^0)^0(\lambda x.x^0)^0)^0 y^0)^0 \to ((\lambda x.x^0)^1 y^0)^0$

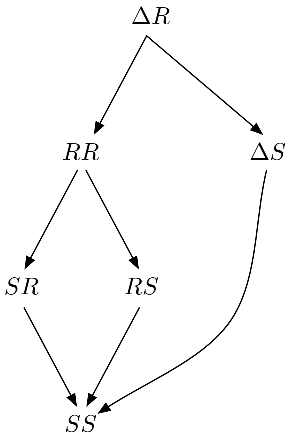
- Let $\Omega = (\lambda x.xx)(\lambda x.xx)$

  Then
  $\Omega_0 = (\Delta^0\Delta^0)^0 \to (\Delta^1\Delta^1)^1 \to (\Delta^2\Delta^2)^2 \to \cdots (\Delta^{60}\Delta^{60})^{60} \to \cdots$
  where $\Delta^n = (\lambda x.(x^0x^0)^0)^n$

- redexes created (degree 0), redexes created by redex(es) created (degree 1), ... chains of creations. [event structures of λ-calculus]

Redex families

Residuals and creation



Let redex R create redex S.

All R redexes are residuals of R redex in initial ΔR.

The S created redexes are not all residuals of a unique S.

But the S redexes are only connected by a zigzag of residuals.

Furthermore the S redexes are created in a "same way" by residuals of a same R-redex.

The historical λ-calculus – 1

- Let ρ be reduction $M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} \cdots \xrightarrow{R_{60}} \cdots \xrightarrow{R_n} M_n$ and let R be a redex in M_n.

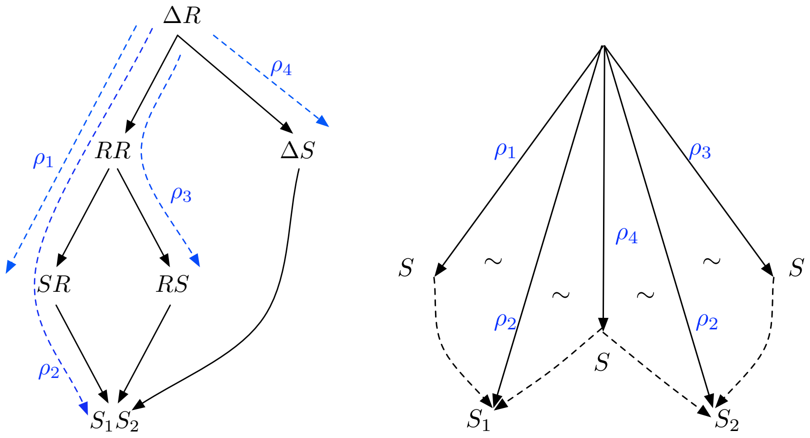
Definition

We write $\langle \rho, R \rangle$ when R is a redex in the final term of ρ. We say R has history ρ.

- The historical redexes $\langle \rho, R \rangle$ and $\langle \sigma, S \rangle$ are in a same family if connected by previous zigzag.

- Histories are considered up to permutation equivalence $\sim$ on reductions.

The historical λ-calculus – 2



$$\langle \rho_1, S \rangle \simeq \langle \rho_2, S_1 \rangle \simeq \langle \rho_4, S \rangle \simeq \langle \rho_2, S_2 \rangle \simeq \langle \rho_3, S \rangle$$

The family equivalence between historical redexes is the symmetric, transitive, reflexive closure or the residual relation.

The historical λ-calculus – 3

families of redexes by naming scheme of the labeled λ-calculus.

	letters	a, b, c
	labels	$\alpha, \beta, \gamma := a \mid \alpha\lceil\beta\rceil\gamma \mid \alpha\lfloor\beta\rfloor\gamma$
	expressions	$M, N := x^\alpha \mid (MN)^\alpha \mid (\lambda x.M)^\alpha$
$(h(\alpha) \leq 60)$	β-conversion	$((\lambda x.M)^\alpha N)^\beta \to \beta \cdot \lceil\alpha\rceil \cdot M\{x := \lfloor\alpha\rfloor \cdot N\}$

concat	$\alpha \cdot x^\beta = x^{\alpha\beta}$	substitution	$x^\alpha\{x := P\} = \alpha \cdot P$
	$\alpha \cdot (MN)^\beta = (MN)^{\alpha\beta}$		$(MN)^\alpha\{x := P\} = (M\{x := P\}N\{x := P\})^\alpha$
	$\alpha \cdot (\lambda x.M)^\beta = (\lambda x.M)^{\alpha\beta}$		$(\lambda y.M)^\alpha\{x := P\} = (\lambda y.M\{x := P\})^\alpha$

Theorem (Church-Rosser+SN)

The labeled (60 bounded) labeled λ-calculus is a canonical systems.

Comes from associativity of concatenation since $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

Generalized Finite Developments

Finite developments revisited

Theorem (generalised finite developments – square lemma)

There are no infinite reductions of a finite set $\mathcal{F}$ of families in the reduction graph of M. All development of $\mathcal{F}$ end on same term.

Proof : obvious since *up-to-N* strongly normalises where N is the maximum degree of redexes in $\mathcal{F}$. For instance $N = 60$.

Theorem (finite developments+ — the cube lemma)

The notion of residuals of reductions (and hence of redex families) is consistent with generalised finite developments.

Corollary

A λ-term is strongly normalizable iff he only can create a finite number of redex families.

Finite chains of families creation

Only 3 cases of redex creations :

1. $(\lambda x. \cdots xN \cdots)(\lambda y.M) \to \cdots (\lambda y.M)N' \cdots$
2. $(\lambda x.(\lambda y.M)N)P \to (\lambda y.M')P$
3. $(\lambda x.x)(\lambda y.M)N \to (\lambda y.M)N$

In 1st-order typed λ-calculus :

1. $(\lambda x. \cdots xN \cdots)^{\alpha \mapsto \beta}(\lambda y.M) \to \cdots (\lambda y.M)^{\alpha} N' \cdots$
2. $(\lambda x.\lambda y.M)^{\alpha \mapsto \beta} NP \to (\lambda y.M')^{\beta} P$
3. $(\lambda x.x)^{\alpha \mapsto \alpha}(\lambda y.M)N \to (\lambda y.M)^{\alpha} N$

In Hyland-Wadsworth's λ-calculus :

1. $(\lambda x. \cdots xN \cdots)^{n+1}(\lambda y.M) \to \cdots (\lambda y.M)^{n} N' \cdots$
2. $((\lambda x.\lambda y.M)^{n+1} N)P \to (\lambda y.M')^{n} P$
3. $(\lambda x.x)^{n+1}(\lambda y.M)N \to (\lambda y.M)^{n} N$

Same in up-to-60 λ-calculus . . . .

Conclusion

Conclusion

- no infinite chain of creations is equivalent to strong normalisation.
- redex families exist also in TRS and many other reduction systems. E.g. redo it as permutation equivalences were treated in the almost everywhere rejected paper [Huet, Lévy 80]
- redo SN without the Tait/Girard reductibility incomprehensible reductibility method (with candidates or not).
- causality in reduction systems correspond to dependency, and can be useful for information flow, security – integrity properties. [Tomasz Blanc 06]
- understand more of the λ-calculus to be able to treat "real complex systems"

Future work

OBJECTIVE