Sharing in the weak lambda-calculus

Tomasz Blanc
Jean-Jacques Lévy
Luc Maranget

INRIA Rocquencourt

Dec 19, 2005

$$(\lambda x.M) \to M[x := N] + \delta M M \to M = \text{non confluent}$$

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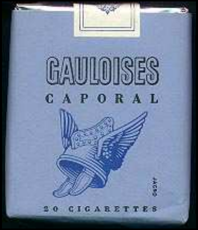
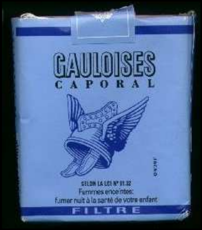
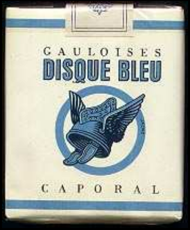
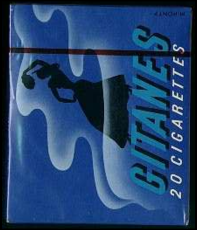
$$(\lambda x.M) \to M[x := N] + \delta MM \to M = \text{non confluent}$$

"Retomber dans la cloppe, JAMAIS ! !"
Et sinon ca va sans cloppe ?
http ://sos-cloppes.over-blog.com/
cette saloperie de clope
Voila comment j'ai réussi à stopper la clop
il clope comme un pompier









t'as pas une clope ?
FUMER UNE CLOPE
ca vaut pas un clope. Des clopes !
des clopinettes.

Clop

Laboratoire d'Analyse et de Traitement Informatique de la Langue Française

Trésor de la Langue Française informatisé (version simplifiée)

Nouvelle recherche

>>

Signification des couleurs		
Mot recherché	Expressions ou locutions	Définitions

CLOP(E), *(CLOP, CLOPE)* subst. masc.

Arg. Mégot de cigare ou de cigarette. *Jeter, ramasser, fumer un clope* :

- Ô mon vieux Maroni, ô Cayenne la douce!
  Je vois les corps penchés de quinze à vingt fagots
  Autour du mino blond qui fume les mégots
  Crachés par les gardiens dans les fleurs et la mousse.
  Un **clop** mouillé suffit à nous désoler tous.
  GENÊT, *Poèmes,* Le condamné à mort, 1948, p. 23.

– *P. ext.* Cigarette. *Le Nantais posa son clope dans le cendrier* (A. LE BRETON, *Razzia sur la chnouf,* 1954, p. 32)

– *Loc. Des clopes.* Rien (*cf.* ESN. 1966).

Étymol. et Hist. Apr. 1900 *clope* « mégot de cigare ou de cigarette » (Notes manuscrites ajoutées sur les feuillets des notes de Nouguier, p. 72); 1925 loc. *des clopes* « rien » (expr. pop. *béqueter des clopes* « jeûner » d'apr. ESN.); 1942 « cigarette » (maquisards d'apr. ESN. : se rouler un **clope**); 1947 *clop* « mégot » (L. STOLLÉ, *Douze récits hist. racontés en arg.,* p. 5). Orig. inconnue (*FEW* t. 21, p. 501; ESN.). **Fréq. abs. littér.** *Clop* : 1.

Plan

1. The weak λ-calculus
2. Properties of the weak λ-calculus
3. Sharing in the λ-calculus
4. Sharing in the weak λ-calculus
5. Sharing of subterms
6. Conclusion

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The weak λ-calculus (1/4)

- λ-calculus without the ξ-rule

$$(\xi)\ \frac{M \to N}{\lambda x.M \to \lambda x.N}$$

- is not confluent

$$\begin{array}{ccc} (\lambda x.\lambda y.M)N & \longrightarrow & (\lambda x.\lambda y.M)N' \\ \downarrow & & \downarrow \\ (\lambda y.M[\![x \backslash N]\!]) & & (\lambda y.M[\![x \backslash N']\!]) \end{array}$$

- Our objectives :
  - find a confluent extention of the weak λ-calculus,
  - re-study standard properties (FD, standardization, etc),
  - find a theory of sharing in this calculus [Wadsworth, Shivers-Wand]

The weak λ-calculus (1/4)

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The weak λ-calculus (2/4)

- weakening the ξ-rule :

$$(\xi')\ \frac{M \xrightarrow{R} N \quad x \notin R}{\lambda x.M \xrightarrow{R} \lambda x.N}$$

  (R is the redex contracted in $M \xrightarrow{R} N$)

- redexes with free variables not bound in M can be contracted
- now

$$\begin{array}{ccc} (\lambda x.\lambda y.M)N & \longrightarrow & (\lambda x.\lambda y.M)N' \\ \downarrow & & \downarrow \\ (\lambda y.M[\![x \backslash N]\!]) & \twoheadrightarrow & (\lambda y.M[\![x \backslash N']\!]) \end{array}$$

The weak λ-calculus (3/4)

- λ-terms

$$M, N ::= x \mid MN \mid \lambda x.M$$

- β-reduction is

$$(\beta) \qquad R = (\lambda x.M)N \xrightarrow{R} M[\![x \backslash N]\!]$$

- Substitution $M[\![x \backslash N]\!]$ defined as usual :

$$\begin{array}{rcl} x[\![x \backslash P]\!] & = & N \\ y[\![x \backslash P]\!] & = & y \\ (MN)[\![x \backslash P]\!] & = & M[\![x \backslash P]\!]\ N[\![x \backslash P]\!] \\ (\lambda y.M)[\![x \backslash P]\!] & = & \lambda y.M[\![x \backslash P]\!] \quad (x \neq y,\ y \notin P) \end{array}$$

The weak λ-calculus (4/4)

- context rules

$$(\nu)\ \frac{M \xrightarrow{R} M'}{MN \xrightarrow{R} M'N} \qquad (\mu)\ \frac{N \xrightarrow{R} N'}{MN \xrightarrow{R} MN'}$$

$$(\xi')\ \frac{M \xrightarrow{R} M' \quad x \notin R}{\lambda x.M \xrightarrow{R} \lambda x.M'}$$

- extra rules
  - unlabelling

$$(w)\ \frac{M \xrightarrow{R} N}{M \rightarrow N}$$

  - $M \twoheadrightarrow N$ for transitive and reflexive closure

Properties of the weak λ-calculus (1/2)

Theorem 1 [Church-Rosser] The weak λ-calculus is confluent.

Proof: Standard Tait–Martin-Lof proof. □

- residuals of disjoint redexes are disjoint.
  - $(\lambda x.Ix)(Jy)$ with $I, J = \lambda x.x$.
  - In strong λ-calculus, the two disjoint Ix and Jy redexes have nested residuals :

$$(\lambda x.Ix)(Jy) \to I(Jy)$$

  - impossible in weak λ-calculus.
- Finite developments theorem is easy to prove.

Properties of the weak λ-calculus (2/2)

- standard reductions

$$M = M_0 \rightarrow M_1 \rightarrow \ldots M_n = N \quad (n \geq 0)$$

$\forall i . \forall j .\ 0 \leq i < j < n$, then R_j is not a residual of a redex internal to or to the left of the R_i.

- We write $M \underset{\text{st}}{\twoheadrightarrow} N$

Theorem 2 [Standardization] If $M \twoheadrightarrow M'$, then $M \underset{\text{st}}{\twoheadrightarrow} M'$.

- Normalization strategy to the “best” normal form (normal reduction is weak until abstractions).

Other theories of weak and strong λ-calculus

- weak explicit substitutions with closures
- Hindley's rule

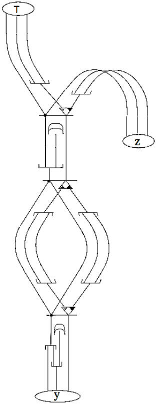
$$(\sigma)\ \frac{N \to N'}{M[\![x \backslash N]\!] \to M[\![x \backslash N']\!]}$$

- computational monads
- Ariola, et al ; Launchbury

- explicit substitutions (not confluent, non normalizable)
- classic λ-calculus (confluent, normalizable, complex theory of sharing)

Sharing in the λ-calculus (1/5)

- difficult in classical λ-calculus
  $\Rightarrow$ interaction nets $+$ geometry of interaction



- not elementary recursive

Sharing in the λ-calculus (2/5)

Take $(\lambda x.k(xa)(xb))(\lambda y.(Iy)) \rightarrow k(\bullet a)(\bullet b)$ where $\bullet = \lambda y.(Iy)$

- in the classical λ-calculus,
  - sharing is complex because of sharing of functions
  - sharing of subcontexts
  - sharing of boxes
- in weak λ-calculus,
  - one cannot contract redexes whose free variables are bound in surrounding context
  - sharing of subterms
  - sharing of trees

Sharing in the λ-calculus (3/5)

- find a confluent theory of sharing
- sharing $=$ labelling
  $\Rightarrow$ find a confluent labelled λ-calculus.

Sharing in the λ-calculus (4/5)

Terms :

$$
\begin{array}{lcll}
U, V & ::= & \alpha : X & \text{labeled term} \\
X, Y & ::= & S \mid U & \text{clipped or labeled term} \\
S, T & ::= & x \mid UV \mid \lambda x.U & \text{clipped term} \\
\alpha, \beta & ::= & a \mid \lceil \alpha' \rceil \mid \lfloor \alpha' \rfloor & \text{labels} \\
\alpha', \beta' & ::= & \alpha_1 \alpha_2 \cdots \alpha_n \quad (n > 0) & \text{compound labels}
\end{array}
$$

Reduction

$$
(\ell) \qquad (\alpha' \cdot \lambda x.U)V \;\rightarrow\; \lceil \alpha' \rceil : U\,[\![x \backslash \lfloor \alpha' \rfloor : V]\!]
$$

where

$$
\alpha_1 \alpha_2 \cdots \alpha_n \cdot S = \alpha_1 : \alpha_2 : \cdots \alpha_n : S
$$

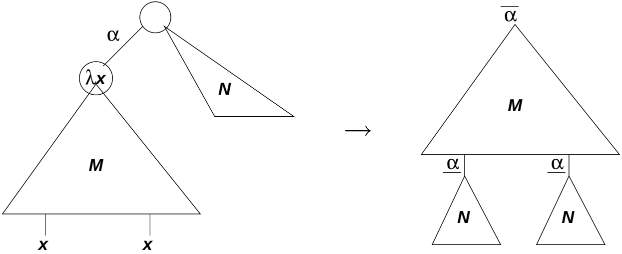
Sharing in the λ-calculus (5/5)

Context rules

$$(\nu)\ \frac{U \to U'}{UV \to U'V} \qquad (\lambda)\ \frac{X \to X'}{\alpha : X \to \alpha : X'}$$

$$(\mu)\ \frac{V \to V'}{UV \to UV'} \qquad (\xi)\ \frac{U \to U'}{\lambda x.U \to \lambda x.U'}$$

Graphically



Sharing in the weak λ-calculus (1/5)

Terms :

$$
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X, Y & ::= & S \mid U & \text{clipped or labeled term} \\
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\alpha, \beta & ::= & a \mid \lceil \alpha' \rceil \mid \lfloor \alpha' \rfloor \mid [\alpha', \beta] \mid \langle \alpha', \beta \rangle & \text{labels} \\
\alpha', \beta' & ::= & \alpha_1 \alpha_2 \cdots \alpha_n \quad (n > 0) & \text{compound labels}
\end{array}
$$

Reduction

$$(\ell) \qquad R = (\alpha' \cdot \lambda x.U)V \xrightarrow{R} \lceil \alpha' \rceil : (\alpha' \otimes U)[\![x \backslash \lfloor \alpha' \rfloor : V]\!]$$

where

$$\alpha_1 \alpha_2 \cdots \alpha_n \cdot S = \alpha_1 : \alpha_2 : \cdots \alpha_n : S$$

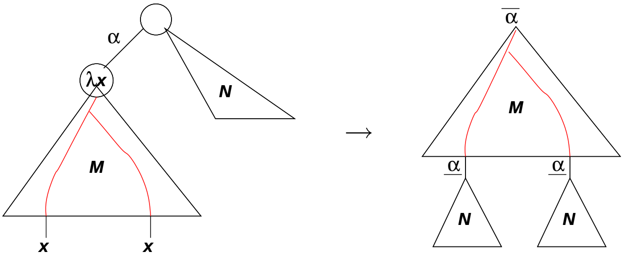
Sharing in the weak λ-calculus (2/5)

Context rules

$$(\nu)\ \frac{U \stackrel{R}{\rightarrow} U'}{UV \stackrel{R}{\rightarrow} U'V} \qquad (\lambda)\ \frac{X \stackrel{R}{\rightarrow} X'}{\alpha : X \stackrel{R}{\rightarrow} \alpha : X'}$$

$$(\mu)\ \frac{V \stackrel{R}{\rightarrow} V'}{UV \stackrel{R}{\rightarrow} UV'} \qquad (\xi')\ \frac{U \stackrel{R}{\rightarrow} U' \quad x \notin R}{\lambda x.U \stackrel{R}{\rightarrow} \lambda x.U'}$$

Graphically



Diffusion

$$
\begin{array}{rcl}
\alpha' \textcircled{x} X & = & X \ \text{ if } x \notin X \\[1em]
\alpha' \textcircled{x} x & = & x \\[1em]
\alpha' \textcircled{x} \lambda y.U & = & \lambda y.\alpha' \textcircled{x} U \ \text{ if } x \in \lambda y.U \\
\alpha' \textcircled{x} \beta : X & = & [\alpha', \beta] : \alpha' \textcircled{x} X \ \text{ if } x \in X \\
 & & \text{tagging} \\[1em]
\alpha' \textcircled{x} UV & = & (\alpha' \textcircled{x} U \ \ \alpha' \textcircled{x} V) \ \text{ if } x \in U \\
\alpha' \textcircled{x} UV & = & (\langle \alpha', U \rangle \ \ \alpha' \textcircled{x} V) \ \text{ if } x \notin U \text{ and } x \in V \\
 & & \text{marking} \\[1em]
\langle \alpha', \beta : X \rangle & = & \langle \alpha', \beta \rangle : X
\end{array}
$$

Sharing in the weak λ-calculus (4/5)

Diffusion in $R = (\alpha' \cdot \lambda x.U)V \xrightarrow{R} \lceil \alpha' \rceil : (\alpha' \, \textcircled{x} \, U)[\![x \backslash \lfloor \alpha' \rfloor : V]\!]$

- "tagging" paths to occurences of free variable x.
- "marking" redexes unleashed by reduction of R.
  - created redexes by contraction of R are tagged or marked by α'. They can also contain $\lceil \alpha' \rceil$ or $\lfloor \alpha' \rfloor$.
  - residual of redexes with name α' are also named α'.
  - "marking" is necessary in following example :
    $R = (\lambda x.Ix)y$, where $I = \lambda u.u$.
    Then Ix is not a redex in $(\lambda x.Ix)y$,
    but it becomes redex Iy after contracting R.

Sharing in the weak λ-calculus (5/5)

Lemma 1 If $X \stackrel{R}{\rightarrow} X'$ and $x \notin R$, then $\alpha' \, \textcircled{x} \, X \rightarrow \alpha' \, \textcircled{x} \, X'$

Lemma 2 If $U \rightarrow U'$, then $X[\![x \backslash U]\!] \twoheadrightarrow X[\![x \backslash U']\!]$

Theorem 3 [Church-Rosser] The weak labeled λ-calculus is confluent.

Proof: By the Tait–Martin-Lof method. $\square$

Sharing of subterms (1/4)

- λ-terms are represented by dags,
- labels represent addresses in dags,
- at beginning no sharing, all addresses of subterms are distinct.

Notation

Init(U) *when every subterm of* U *is labeled with a distinct letter (a, b, c, ...).*

Invariant 1 $\mathcal{P}(W)$ holds iff, for any couple of subterms $\alpha : X$ and $\beta : Y$ such that $\alpha \simeq \beta$, we have $X = Y$.

Theorem 4 Let *Init*(U) and $U \Longrightarrow V$, then $\mathcal{P}(V)$.

Sharing of subterms (2/4)

- $\alpha \simeq \beta$ when $\alpha = \beta$ up to marking
- $U \overset{\alpha'}{\Longrightarrow} V$ when all redexes of name α' are contracted in U, result is V.

where

$$
\begin{array}{ll}
a \simeq a & \\
\lceil \alpha' \rceil \simeq \lceil \alpha' \rceil & \lfloor \alpha' \rfloor \simeq \lfloor \alpha' \rfloor \\
\beta \simeq \gamma \;\Rightarrow\; [\alpha', \beta] \simeq [\alpha', \gamma] & \beta \simeq \gamma \;\Rightarrow\; \langle \alpha', \beta \rangle \simeq \langle \alpha', \gamma \rangle \\
\beta \simeq \gamma \;\Rightarrow\; \beta \simeq \langle \alpha', \gamma \rangle & \beta \simeq \gamma \;\Rightarrow\; \langle \alpha', \beta \rangle \simeq \gamma
\end{array}
$$

Lemma 3 If $X \xrightarrow{R} Y$ and redex S in Y is created by this reduction step, then $\mathit{name}(R) \prec \mathit{name}(S)$.

$$
\alpha' \prec \lceil \alpha' \rceil \qquad \alpha' \prec \lfloor \alpha' \rfloor \qquad \alpha' \prec [\alpha', \beta] \qquad \alpha' \prec \langle \alpha', \beta \rangle
$$

$$
\alpha' \prec \beta_i \;\Rightarrow\; \alpha' \prec \beta_1 \ldots \beta_n \qquad \alpha' \prec \beta' \prec \gamma' \;\Rightarrow\; \alpha' \prec \gamma'
$$

Sharing of subterms (3/4)

- interesting proof, with 4 invariants

Invariant 2 $\mathcal{Q}(W)$ holds iff we have $\alpha' \not\leq \beta$ for every redex R with name α' and any subterm $\beta : X$ in W.
Invariant 3 $\mathcal{R}(W)$ holds iff for any clipped subterm UV in W, we have either $U = a : X$, or $U = [\alpha', \beta] : X$, or $U = \langle \alpha', \beta \rangle : X$.
Invariant 4 $\mathcal{S}(W)$ holds iff, for any application subterms $\beta : (\alpha : X)U$ and $\gamma : (\alpha : Y)V$, we have $\beta \simeq \gamma$.

Lemma 4 If $\mathcal{Q}(W)$ and $W \stackrel{\gamma'}{\Longrightarrow} W'$, then $\mathcal{Q}(W')$.
Lemma 5 If $\mathcal{R}(W)$ and $W \rightarrow W'$, then $\mathcal{R}(W')$.

Lemma 6 If $\mathcal{P}(W) \wedge \mathcal{Q}(W) \wedge \mathcal{R}(W) \wedge \mathcal{S}(W)$ and $W \stackrel{\gamma'}{\Longrightarrow} W'$, then $\mathcal{S}(W')$.

Lemma 7 If $\mathcal{P}(W) \wedge \mathcal{Q}(W) \wedge \mathcal{R}(W) \wedge \mathcal{S}(W)$ and $W \stackrel{\gamma'}{\Longrightarrow} W'$, then $\mathcal{P}(W')$.

Sharing of subterms (4/4)

- labeled λ-calculus corresponds to Wadsworth's phD (ch.4) 2nd method
  - diffusion $=$ copying
  - labels $=$ adresses
  - calculus is confluent
- how to check $x \in U$ efficiently ?
  - Shivers-Wand's method (bottom-up copying from bound variables to root of function bodies.
  - our method models slightly more shared strategy since not recursively copying binders met on path to root of function bodies.
  - compiling this check is not easy since sets of variables may change during computation.
  - similar to full lazyness (PJ, Hugues), but without super combinators.

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Conclusion

- re-do all theory of optimal reductions,
- links with supercombinators and other compiler techniques,
- weak λ-calculus desserves a theory
- theory simpler than for TRS

Open problems

OBJECTIVE

Final remarks