An abstract standardisation theorem

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Abstract

The standardisation theorem is a key theorem in the λ-calculus. It implies that any normal form can be reached by the normal order (leftmost outermost) strategy. The theorem states that any reduction may be rearranged in a top-down and left-to-right order. This also holds in orthogonal term rewriting systems (TRS), although the left-to-right order is more subtle. We give a new presentation of the standardisation property by means of four axioms about the residual and nesting relations on redexes. This axiomatic approach provides a better understanding of standardisation, and makes it applicable in other settings, such as dags or interaction networks. We also treat conflicts between redexes (critical pairs in TRS). The axioms include Berry's stability, proving it to be a intrinsic notion of *deterministic* calculi.

1 Introduction

The λ-calculus has two main syntactic theorems. One is the Church-Rosser theorem, which induces uniqueness of normal forms. The second one is the standardisation theorem [1], which shows that leftmost outermost reduction is a terminating strategy. These two fundamental theorems can be found in many different situations, such as in PCF (typed λ-calculus augmented with several δ-rules, such as recursion or arithmetic), in orthogonal term rewriting systems [2], in orthogonal dags and in interaction networks. These different settings must share a common property to yield these two results. As proving these properties time and again is rather frustrating, there have been attempts to define an axiomatic version of the Church-Rosser property [3]. However, to our knowledge, nothing has been done for the standardisation theorem. We try here to fill this gap, and to present simple axioms, that we feel are the essence of the standardisation property.

2 The standardisation property

In the λ-calculus, the normal order is a terminating strategy. For instance, take $K = \lambda xy.x$, $\Delta = \lambda x.xx$. Then the normal reduction

$$Ka(\Delta\Delta) \rightarrow (\lambda y.a)(\Delta\Delta) \rightarrow a$$

reaches the normal form a, while the rightmost reduction loops. This is a corollary of the standardisation theorem, which also works for non-normal forms. A reduction

$$M_0 \stackrel{R_1}{\rightarrow} M_1 \stackrel{R_2}{\rightarrow} M_2 \ldots \stackrel{R_n}{\rightarrow} M_n$$

is *standard*, when for every i, j, such that $i < j$, the redex R_j is not a copy (residual along this reduction) of a redex R'_i outer to or to the left of R_i in M_{i-1}. Clearly, leftmost outermost reductions are standard, but so are further reductions, like

$$I(Ka(\Delta\Delta)) \rightarrow I((\lambda y.a)(\Delta\Delta)) \rightarrow Ia$$

where $I = \lambda x.x$. In a standard reduction, when a redex is contracted, all redexes to the left of it or outer to it are frozen. Hence standard reductions work top-down and left-to-right. The standardisation theorem [1] states that, if $M \rightarrow^* N$, there is a standard reduction from M to N. The proof [4] is not easy, but heavily relies on the *left-linearity* of the λ-calculus.

It is rather easy to understand the outside-in part of the standard reduction, but left-to-right is less natural. In our examples, the righmost outermost reductions loop. Intuitively, redexes may be created to the right, as in $(\lambda y.a)(\Delta\Delta)$, and a looping redex may be nested in a K-redex.

But left-to-right is no longer valid with δ-rules. Consider the case of first order term rewriting systems (TRS) [2], and, for instance, the following system $\{A \rightarrow A, B \rightarrow C, F(x, C) \rightarrow D\}$. Then

$$F(A, B) \rightarrow F(A, C) \rightarrow D$$

reaches the normal form, whereas the leftmost reduction loops. Here terminating strategies are clearly rightmost, and we suspect that we may define a notion of normal reduction, representing the minimum necessary to reach the normal form. However, strategies may be subtle, since some operators may solicit rightmost reductions, others leftmost, while others may change strategy depending on the values of their arguments. In [2], a complicated definition of standard reduction is given for *orthogonal* TRS, i.e., left-linear and without critical pairs, and the standardisation theorem is proved. This definition is very technical because it involves tracing (residuals) of any subexpression in terms, and the structure of computed objects is mixed with the nature of transitions (redexes).

Remark that it is incorrect to simply drop the left-to-right or right-to-left part of the definition of a standard reduction, since then

$$Ka(\Delta\Delta) \to Ka(\Delta\Delta) \to (\lambda y.a)(\Delta\Delta) \to a$$

or

$$F(A,B) \to F(A,B) \to F(A,C) \to D$$

would be standard reductions. But, in these two reductions, the first step may be permuted with the following one, which gives

$$Ka(\Delta\Delta) \to (\lambda y.a)(\Delta\Delta) \to (\lambda y.a)(\Delta\Delta) \to a$$

or

$$F(A,B) \to F(A,C) \to F(A,C) \to D$$

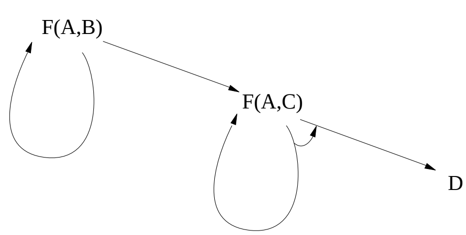
showing a conflict between steps 2 and 3 since, in the permuted reductions, the third contracted redex is a copy (residual) of a redex external to the second contracted one. So, looking backwards is not enough for redexes. It is also necessary to search forwards on permuted reductions.

Also, note that we did not consider arbitrary permutation of reductions. We have considered permutation of disjoint (non-nested) redexes. This remark is technical, and simplifies the introduction to our setting, but will be proved unnecessary.

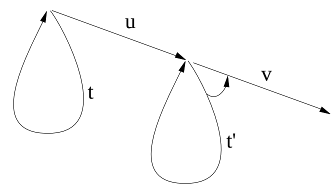
Thus it seems enough to keep trace of redexes (through residuals) and to have a notion of nesting of redexes in each term.

3 Abstract reductions systems

An abstract reduction system is a (possibly infinite) multi-graph, where vertices are terms, and arcs are labeled by redexes. Take, for example, the following TRS



The structure of terms may be ignored. One just needs to keep name for redex occurrences and a partial ordering between outgoing arcs of a given vertex, which will represent the nesting property.



To trace redexes, each arc u induces a *residual relation* $[\![u]\!]$ between an arc t with same origin as u and an arc t' outgoing from the target of u. An arc t', such that $t[\![u]\!]t'$, is a *residual* of t by u. A redex may only be residual of a single redex, i.e., the inverse of $[\![u]\!]$ is a partial function. Moreover, there is no t' such that $t[\![t]\!]t'$, i.e., redexes vanish when contracted.

In the previous example, the only residual relations that hold are the the following: $u[\![t]\!]u$, $v[\![t']\!]v$, $t[\![u]\!]t'$. Moreover, the only nesting relation is $v < t'$, which is graphically represented by marking the corresponding angle. Note that the formalism embraces both the examples of the λ-calculus and of TRS.

So, we represent an abstract reduction systems by a multi-graph, a nesting ordering at each node, and a residual relation for each arc. Now all we do is state the propertiesneeded to define standard reductions and prove the standardisation theorem.

Firstly, we restrict ourselves to left-linear systems (only one occurrence of each meta variable on left hand side of reduction rules) without incompatibilities (critical pairs). This is simply stated by the two following axioms. The first axiom provides Church-Rosser.

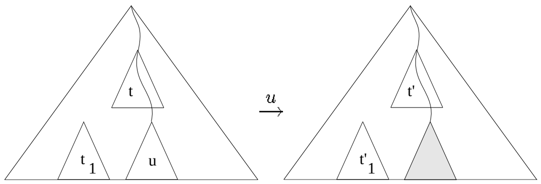
Axiom 0 (Finite Developments) *Let T be a set of redexes in a given term M. A reduction relative to T contracts only residuals of redexes in T. A development of T is a maximal reduction relative to T. We assume that:*

1. *there is no infinite reduction relative to T,*
2. *all developments of T end on the same term,*
3. *if u is any redex in M, the residuals of u do not depend upon the development of F.*

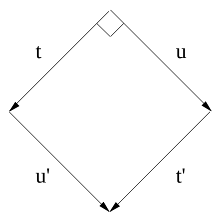
This ensures Church-Rosser, but we do not stress this property here. See [3, 5, 1, 6, 7, 8] for a careful study. Finite Developments will just be useful technically. The first important axiom, which allows us to define standard reductions, is

Axiom 1 (Linearity) $u \not\leq t \Longrightarrow \exists! t'.\ t[\![u]\!]t'$

Thus redexes may disappear, or may be duplicated, only if they are inside contracted redexes. We represent this graphically, in the example of TRS as



Standard reductions can now be defined as outlined at the end of section 2. Let $t \perp u$ mean that t and u are incomparable w.r.t. the nesting ordering, i.e., $t \not\leq u$ and $u \not\leq t$. Graphically the corresponding angle is marked as perpendicular. Now, by combining axiom 0 and axiom 1, it is easy to show that when $t \perp u$ there are two unique redexes t' and u' such that $t[\![u]\!]t'$, $u[\![t]\!]u'$; this gives the following commuting diagram

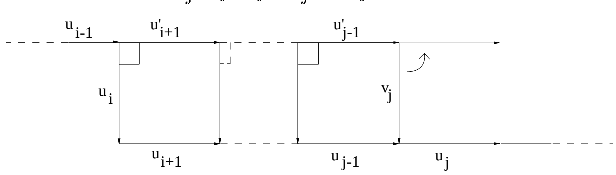


We call this elementary permutation the *square permutation*.

Definition 1 *The reduction* $\xrightarrow{u_1}\xrightarrow{u_2} \dots \xrightarrow{u_1}$ *is standard if for every* i, j, *such that* $i < j$, *there is no reduction*

$$\xrightarrow{u_{i-1}}\xrightarrow{u'_{i+1}}\xrightarrow{u'_{i+2}} \dots \xrightarrow{u'_{j-1}}\xrightarrow{v_j}\xrightarrow{u_j}$$

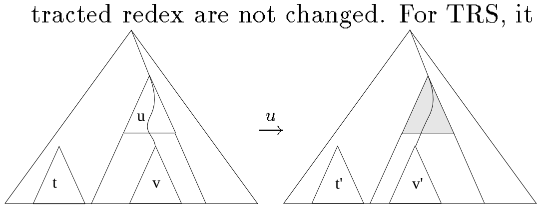
such that $v_{i+1} = u_i$, $v_k \perp u'_k$, $u'_k[\![v_k]\!]u_k$, $v_k[\![u'_k]\!]v_{k+1}$ *for* $i < k < j$, *and* $u'_j[\![v_j]\!]u_j$, $u'_j \leq v_j$.



To prove the standardisation theorem, we make use of a second axiom.

Axiom 2 (Context free)
$(u \not\leq t,\ t[\![u]\!]t',\ v[\![u]\!]v') \Longrightarrow (t \Re v \Leftrightarrow t' \Re v')$ *where* $\Re$ *is* $<$ *or* $>$.

This axiom means that locations outside the contracted redex are not changed. For TRS, it means



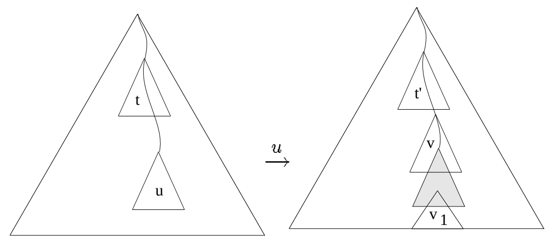
Theorem 1 *If* $M \to^* N$, *there is a standard reduction from* M *to* N.

Proof: by Klop's method [4]. See section 5 for a detailed proof.

Now, a third axiom makes our definition of standard reduction more natural.

Axiom 3 (Creation)
$(t < u,\ t[\![u]\!]t',\ \not\exists v'.\ v'[\![u]\!]v) \Longrightarrow t' < v$

This axiom, which appeared in [9], states that it is impossible to create a redex through another one. This is due to the fact that our reduction systems are conflict free (critical pairs for TRS). Graphically,



The proposition below shows that standard is indeed outside-in.

Proposition 1 *If a redex* x *contains the first redex to be reduced by a standard derivation* d, *then no residual of* x *is nested into one redex computed by* d.

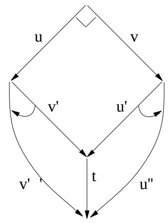
This third axiom also enables us to give a more natural, but equivalent, definition of a standard reduction.

Proposition 2 *The definition of a standard reduction is equivalent if we replace* $u'_k \perp v_k$ *by* $u'_k \not\leq v_k$.

4 Uniqueness of standard reductions

In the λ-calculus, there is only one standard reduction to the normal form. However, there may several standard reductions between two arbitrary terms. Take $I = \lambda x.x$. Then $I(Ix) \to Ix$ is computed by contracting any of the two redexes, and a single step reduction is always standard. It is analogous to ambiguous context free languages where a word may be obtained by two different parse trees. However, a parse tree is totally defined by its leftmost derivation. In the λ-calculus, it is hard to speak of parse trees, but a permutation equivalence may be defined on reductions [6, 10, 2, 11], and there is a single standard reduction in each equivalence class.

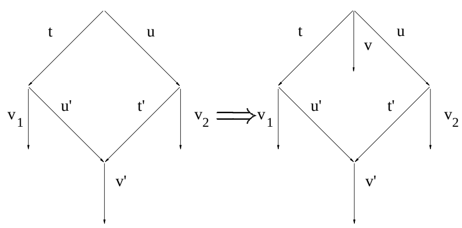
The example $I(Ix)$ is rather simply treated with permutations of reductions. A more contrived example is the *parallel-or* of the reduction system $\{A \to T, por(T, x) \to T, por(x, T) \to T\}$. The following abstract reduction system, which validates the 3 axioms, abstracts the parallel-or example on $por(A, A)$



where there are two standard reductions $\overset{u}{\to}\overset{v''}{\to}$ and $\overset{v}{\to}\overset{u''}{\to}$, equivalent by permutations. To recover uniqueness of standard reductions, we must eliminate this counter example.

Axiom 4 (Stability)
$(t \perp u,\ t[\![u]\!]t',\ u[\![t]\!]u',\ v_1[\![u']\!]v',\ v_2[\![t']\!]v')$
$\Longrightarrow \exists v.\ (v[\![t]\!]v_1,\ v[\![u]\!]v_2)$

This axiom may seem incredibly complicated. It appears quite naturally in the uniqueness proof. It means that the characteristic function of a redex t is stable in the sense of [12]. Graphically,



There is an extra problem; the nesting ordering is not total. There is no notion of left-to-right or right-to-left. So it is hopeless to achieve a unique standard reduction in a permutation class, but standard reductions will be unique up to a square-equivalence. Say that reductions $d = \overset{u_1}{\to}\overset{u_2}{\to} \dots \overset{u_n}{\to}$ and $e = \overset{v_1}{\to}\overset{v_2}{\to} \dots \overset{v_n}{\to}$ are square equivalent ($d \simeq_\perp e$) if e may be obtained from d by a sequence of square permutations.

Proposition 3 *Standard reductions are closed w.r.t to square equivalence.*

This last property shows that a standard derivation is a derivation where no 'standardizing' permutation, that is a permutation commuting the nesting redex before the nested redex, is obtainable after as many square permutation as one desires. Therefore, standard is also a notion of minimality.

Theorem 2 *In each equivalence class, w.r.t. general permutations, there is a unique standard reduction, up to the square equivalence.*

5 Constructions and proofs

5.1 Introduction

This section is devoted to the proofs and constructions underlying the paper. We will introduce technical notations and extension to have proofs more readable.

First $[\![u]\!]$ is extended by induction to $[\![d]\!]$ where d is a derivation: $v[\![ud]\!]w$ iff $\exists v', v[\![u]\!]v'$ and $v'[\![d]\!]w$.

Secondly, f a development of $\mathbf{U}$ will be noted $f \propto \mathbf{U}$, and this notation is extended to $f \propto d[\![e]\!]$ by: $f \propto ud[\![e]\!]$ if $f = f_1 f_2, f_1 \propto u[\![e]\!], f_2 \propto d[\![e]\!]$.

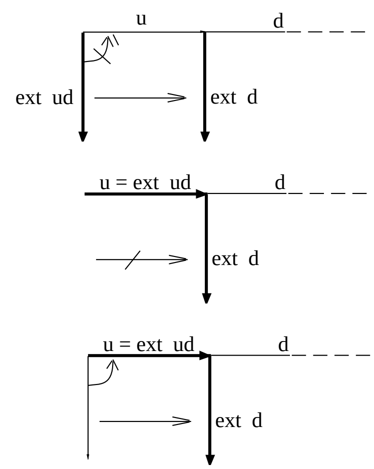
Finally, since $f, f' \propto \mathbf{U}$ implies that $[\![f]\!] = [\![f']\!]$ when $\mathbf{U}$ is a redex verifying finite developments, $[\![\mathbf{U}]\!]$ can be defined as $[\![f]\!]$ with f any development of $\mathbf{U}$.

We will also often write $u[\![v]\!]$ to express $\{w/u[\![v]\!]w\}$ and $[\![v]\!]w$ for $\{u/u[\![v]\!]w\}$, as it is usually done for relations; and $x = u[\![v]\!]$ means that $u[\![v]\!] = \{x\}$.

5.2 The standardization algorithm

We introduce a non deterministic algorithm called **STD**. Given a derivation from M to N, it computes a standard derivation linking the same terms. To build this algorithm, one needs to locate an outermost redex contracted into a derivation d to standardize.

$$ext\ u = u$$
$$ext\ ud = \begin{cases} v & \text{if } \exists v, v[\![u]\!]ext\ d \text{ and } u \not\leq v \\ u & \text{otherwise.} \end{cases}$$



The algorithm grabs this external redex from the derivation, and repeats the process untill there is nothing left:

STD computes step by step the sequence $\overset{v_1}{\to}\overset{v_2}{\to} \dots \overset{v_n}{\to} \dots$ as follows:
let $v_1 = ext\ d$, and choose $d_2 \propto d[\![v_1]\!]$. ... let $v_i = ext\ d_i$, and choose $d_{i+1} \propto d_i[\![v_i]\!]$. ... ... STOP if $d_i = \emptyset$

One should notice that at any step i of the algorithm $(v_1 \ldots v_i)d_{i+1}$ reduces M to N.

One needs a lemma about *ext* and $\propto$ before starting the termination proof:

Lemma 1 *Let $x = ext\ fd$ with $f \propto \mathbf{U}$. Then either x is a redex in $\mathbf{U}$, or $\forall u \in \mathbf{U}, u \not\leq x$ and $x[\![\mathbf{U}]\!]ext\ d$.*

Proof: Let $f = uf'$, with $u \in \mathbf{U}$ and $f' \propto \mathbf{U}[\![u]\!]$. Suppose the lemma is true for $\mathbf{U}[\![u]\!]$: If $x \in \mathbf{U}$, finished. If $x \notin \mathbf{U}$ then $u \not\leq x$ since u is not the external. One verifies that $x[\![u]\!] = ext\ f'd$ is not in $\mathbf{U}[\![u]\!]$ because two redexes cannot have the same residual. Therefore $x[\![f]\!] = ext\ d$ and no redex in $\mathbf{U}[\![u]\!]$ nests $ext\ f'd$. Suppose there is a redex $v \in \mathbf{U}$ such that $v < x$; then by axioms 1 and 2 $v[\![u]\!] < x[\![u]\!]$. Since $v[\![u]\!] \in \mathbf{U}[\![u]\!]$ and $x[\![u]\!] = ext\ f'd$, this is impossible: $\forall v \in \mathbf{U}, v \not\leq x$, and $x[\![f]\!] = ext\ d$. The lemma follows by induction on the finite development. $\square$

Proposition 4 ***STD*** *terminates.*

Proof: We use an induction on the length of d. First if d is a single redex **STD** stops after one step. Suppose **STD** terminates on d, and returns $f = \stackrel{w_1}{\longrightarrow} \ldots \stackrel{w_q}{\longrightarrow}$. Let $e = (\stackrel{v_1}{\rightarrow}\stackrel{v_2}{\rightarrow} \ldots \stackrel{v_i}{\rightarrow} \ldots)$ be the sequence produced by ud. Suppose this sequence is not finite. Let $\mathbf{U}_1 = u$ and $\forall i \geq 1, \mathbf{U}_{i+1} = \mathbf{U}_i[\![v_i]\!]$. Then for any $i \geq 1$, from the lemma one deduces that either v_i is a redex in $\mathbf{U}_i$, or there exists j such that $w_j = v_i[\![\mathbf{U}_i]\!]$. Note that the last case can only happen a finite number of times, though j is counting the times this case happens, and is strictly bounded by q. There is a rank i_0 from which the first case is the only case to occur. But this is refuted by the finite development axiom, since any of these redexes comes from the same set $\mathbf{U}_{i_0}$. Thus **STD** must terminate: the sequence obtained is always a true derivation. $\square$

Proposition 5 ***STD*** *standardizes*

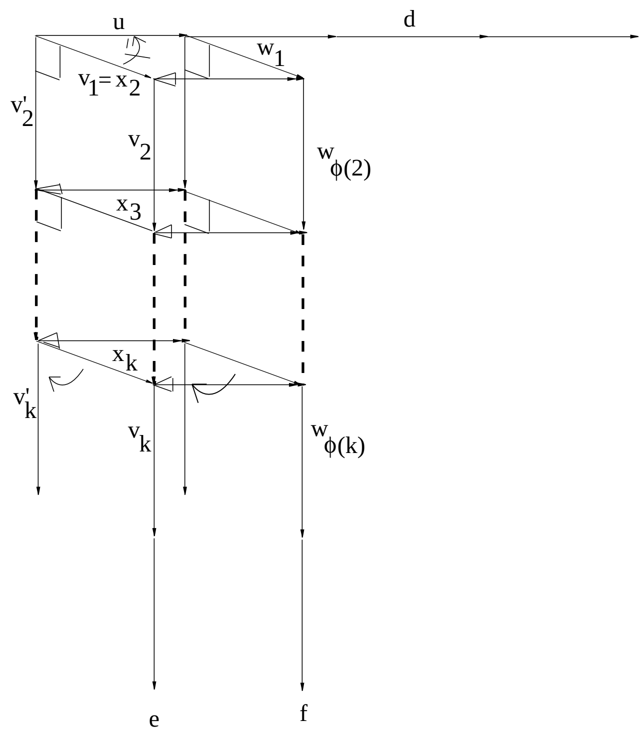
Proof: Now one wants to show that this generated derivation is standard, that is to say it contains no conflict. This proof by induction on the length of d, showing that the first redex in e cannot come into a squareconflict.

Let $e = \stackrel{v_1}{\rightarrow}\stackrel{v_2}{\rightarrow} \ldots \stackrel{v_q}{\rightarrow}$ be what **STD** computes when given ud as a datum, and $f = \stackrel{w_1}{\rightarrow}\stackrel{w_2}{\rightarrow} \ldots \stackrel{w_p}{\rightarrow}$ the result it would have obtained making the same choices than to obtain e, that is to say $e[\![u]\!] = f$.

We want to prove that a conflict between v_1 and v_k implies a conflict into f. By hypothesis, f is standard, Suppose there is a conflict between v_1 and v_k in e.

If $v_1 = u$, since $e = uf$, there would exist a redex w' such that $w'[\![u]\!]w_1$ and $w' \perp u$ or $w' < u$ from the definition of a conflict. Since $w_1 = ext\ d$, w' would then be $ext\ ud = u$. Therefore, if $v_1 = u$, no conflict involves v_1.

Otherwise, $u \not\leq v_1$, and thus $v_1[\![u]\!]w_1$. By a sequence of square permutations, v_1 has a residual x_k nested in v'_k, and v_k a residual of v'_k by x_k. $x_2 = v_1$ and $\forall i \in [2..k], v'_i[\![x_i]\!]v_i, x_{i+1} = x_i[\![v'_i]\!]$; with $\forall i \in [1..k-1], v'_i \perp x_i$. One defines $\mathbf{U}_1 \triangleq u$, $\mathbf{U}_{i+1} \triangleq \mathbf{U}_i[\![v_i]\!]$, and $\mathbf{U}'_2 \triangleq u$, $\mathbf{U}'_{i+1} \triangleq \mathbf{U}'_i[\![v'_i]\!]$. Therefore, by finite development: $\mathbf{U}'_i[\![x_i]\!] = \mathbf{U}_i$, and one verifies easily by axiom 2 by the sequence of v'_i that $\forall i \in [2,k], \forall t \in \mathbf{U}'_i, t \not\leq x_i$.

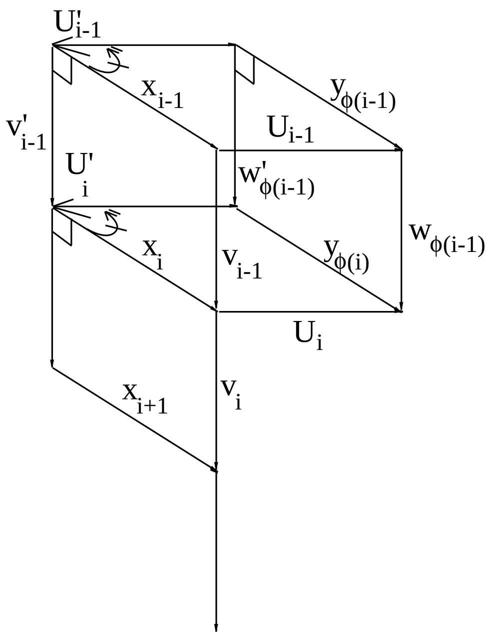


Suppose one built $\forall j < i$, $\phi(j), w'_{\phi(j)}$ and $y_{\phi(j)+1}$ such that $v'_2 \ldots v'_j[\![u]\!]w'_2 \ldots w'_{\phi(j)}$, $x_j[\![\mathbf{U}'_j]\!]y_{\phi(j)}$, and w square permutates with $w'_2 \ldots w'_{\phi(j)}$. One easily verifies by finite developments that $y_2 = w$, $\forall j \in [2..\phi(i-1)], w'_j[\![y_j]\!]w_j, y_{j+1} = y_j[\![w'_j]\!]$.

if $i \leq k-1$, if $v'_i[\![\mathbf{U}'_i]\!] = \emptyset$, since $x_i[\![\mathbf{U}'_i]\!] = y_{\phi(i-1)}$ and $v'_2 \ldots v'_i[\![u]\!]w'_2 \ldots w'_{\phi(i-1)}$, one defines $\phi(i) = \phi(i-1)$, and the *i*step is completed. If $v'_i[\![\mathbf{U}'_i]\!] \neq \emptyset$, let us define $\phi(i)$ as $\phi(i-1)+1$. Since two redexes never have the same residual, $v'_i \notin \mathbf{U}'_i$ implies that $v_i \notin \mathbf{U}_i$. Since v_i is defined as the external of a derivation beginning by a development of $\mathbf{U}_i$, by the lemma above, $\forall t \in \mathbf{U}_i, t \not\leq v_i$ and $v_i[\![\mathbf{U}_i]\!]w_\phi(i)$. One notes that $\forall t \in \mathbf{U}'_i, t \not\leq v'_i$ by axiom 2. One defines $w'_\phi(i) = v_i[\![\mathbf{U}'_i]\!]$ and notices that $w'_{\phi(i)}[\![y_{\phi(i)}]\!] = w_{\phi(i)}$. Finally, by axiom 2, since $v'_i \perp x_i$, $w'_{\phi(i)} \perp y_{\phi(i)}$ and one defines $y_{\phi(i)+1} = y_{\phi(i)}[\![w'_{\phi(i)}]\!]$.

if $i = k$, since $v'_k < x_k$ and $\forall t \in \mathbf{U}'_k, t \not\leq x_k$, by transitivity $\forall t \in \mathbf{U}'_k, t \not\leq v'_k$. One defines $\phi(k) = \phi(k-$

$1) + 1$ and $v'_k[\![\mathbf{U}'_k]\!]w'_{\phi(k)}$ with by finite developments: $w'_{\phi(k)}[\![y_{\phi(k-1)}]\!]w_{\phi(k)}$ and by axiom 2: $w'_{\phi(k)} < y_{\phi(k)}$. There is a squareconflict between w_1 and $w_{\phi(k)}$. □



The existence theorem is now a corollary of our construction:

Theorem 1 *If $M \rightarrow^* N$, there is a standard reduction from M to N.*

5.3 Unicity proof

Proof of Proposition 1

By induction on the length of d. Let $d = u_1 \dots u_n$, and $z_1 \triangleq z$.

if $n = 2$, there exists a unique residual z_2 such that $z_1[\![u_1]\!]z_2$. Suppose $u_2 \leq z_2$. By axiom 3 , there exists a redex u'_2 such that $u'_2[\![u_1]\!]u_2$, and it is unique by left-linearity. One checks by axiom 2 that if $u_2 < z_2$ then $u'_2 < z_1$ and by transitivity $u'_2 < u_1$, and if $u_2 = z_2$ then $u'_2 = z_1 < u_1$, both leading to an impossible squareconflict in d. Finally, $u_2 \not\leq z_2$, the lemma is proved for $n = 2$, and by axiom 1, one defines $z_1[\![d]\!] = \{z_3\}$.

let $d = \stackrel{u_1}{\rightarrow}\stackrel{u_2}{\rightarrow} \dots \stackrel{u_{n+1}}{\rightarrow}$ be standard, and $z_1 < u_1$. z_2 is easily defined as $z_1[\![u_1]\!]z_2$. If $z_2 < u_2$ then by induction, no residual of z_1 is ever contained into a redex reduced into d. because $\stackrel{u_2}{\rightarrow}\stackrel{u_3}{\rightarrow} \dots \stackrel{u_{n+1}}{\rightarrow}$ is also standard. Since $u_1 u_2$ is standard, one knows from above that u_2 cannot nest z_2. The last case to study is therefore $u_2 \perp z_2$. By axiom 3 there exists a redex u'_2 such that $u'_2[\![u_1]\!]u_2$. Since d is standard, $u'_2 \not\leq u_1$. By axiom 2, u'_2 is also disjoint to z_1, by transitivity $u_1 \not\leq u'_2$. Mixing both results, one obtains that $u_1 \perp u'_2$, and notices that there is a unique residual u'_1 such that $u_1[\![u'_2]\!]u'_1$. The same for z_1, there exists a unique residual z'_1 of z_1 by u'_2. Finally, one checks by axiom 2 that $z'_1 < u'_1$ and $\stackrel{u'_1}{\rightarrow}\stackrel{u_3}{\rightarrow} \dots \stackrel{u_{n+1}}{\rightarrow}$ is still standard. Thus by induction, no residual of z'_1 is ever contained into redex reduced by $u'_1 u_3 \dots u_{n+1}$, therefore since nor z_1 nor z_2 is contained into u_1 or u_2 respectively, and $z_1[\![u_1 u_2]\!] = z'_1[\![u'_1]\!]$, no residual of z_1 is ever contained in one reduced by d. □

Proof of Proposition 2

d is not standard with our last definition, it cannot be with the new one, since a conflict with square permutations is also a conflict with generalized ones. To prove the equivalence, one has to prove that a generalized conflict in a derivation implies one with square permutations. Let $d = \stackrel{u_1}{\rightarrow} \dots \stackrel{u_n}{\rightarrow}$ be a derivation where there is a generalized conflict between u_k and u_l ($k < l$). After any permutation, one knows by axiom 1 that v_k has only one residual. If no redex in $\stackrel{u_{k+1}}{\rightarrow} \dots \stackrel{u_{l-1}}{\rightarrow}$ is nested by this residual, then any of these is disjoint to the residuals of v_k, and therefore the generalized conflict is only a square conflict. Otherwise, let k' be the first indice such that the residual of u_k nests u'_k. Then if $d' = \stackrel{u'_k}{\rightarrow} \dots \stackrel{u_l}{\rightarrow}$ was standard, no redex reduced by d' could nest a residual of v_k. But this happens on indice l: d' is therefore not standard, d' neither. □

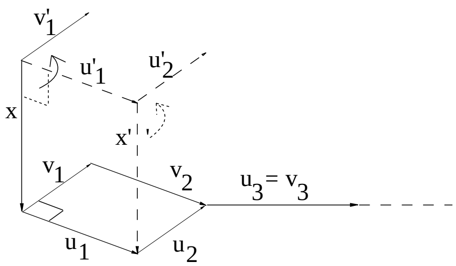
Proof of Proposition 3

We want to show that a standard derivation remains standard after a square permutation. This will be proved by an induction on the length of the derivation, since the proposition is clear for a two redexes derivation. Suppose proved the property for derivations with a length less than n, and let $d = \stackrel{u_0}{\rightarrow}\stackrel{u_1}{\rightarrow} \dots \stackrel{u_n}{\rightarrow}$ be a standard derivation. Suppose that $e = \stackrel{v_0}{\rightarrow}\stackrel{v_1}{\rightarrow} \dots \stackrel{v_n}{\rightarrow}$ obtained from d by permuting the (u_i, u_{i+1})square contains a square conflict between v_k and v_l (with $k < l$). Many cases are straightly impossible using induction or the definition of a square conflict: if $(i = 0, k \geq 2)$, or $(i \geq 1, k \geq 1)$ by induction with $u_1 \dots u_n$, if $(i = 0, k = 0)$ or $(i = 0, k = 1)$ by definition of a squareconflict. if $(i \geq 2, k = 0)$ by induction: suppose there is a squareconflict between u_0 and v_l in e, with $l \geq 2$ otherwise the conflict existed in d, there exists u'_1 such that $u'_1 \perp u_0$; therefore with u'_0 the unique residual of u_0 by u'_1, there is a conflict in $\stackrel{u'_0}{\rightarrow}\stackrel{v_2}{\rightarrow} \dots \stackrel{v_n}{\rightarrow}$, and therefore by induction in $\stackrel{u'_0}{\rightarrow}\stackrel{u_2}{\rightarrow} \dots \stackrel{u_n}{\rightarrow}$. This refutes that d is standard, and proves this case cannot hold.

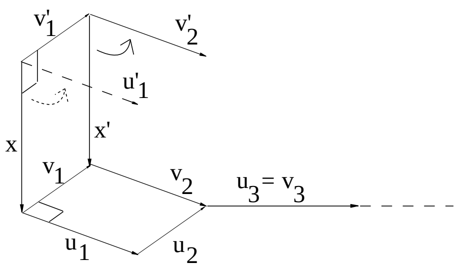
The last case is a bit more complicated to check: $k = 0$ and $i = 1$. Let us call from now $x = u_0 = v_0$.

if $l = 1$, there exists v'_1 such that $v'_1[\![x]\!]v_1$ and $v'_1 < x$. By axiom 3, there exists u'_1 such that $u'_1[\![x]\!]u_1$, otherwise v_1 would nest u_1. If x nests u'_1 then by transitivity v'_1 nests u'_1 and therefore by axiom 2 v_1

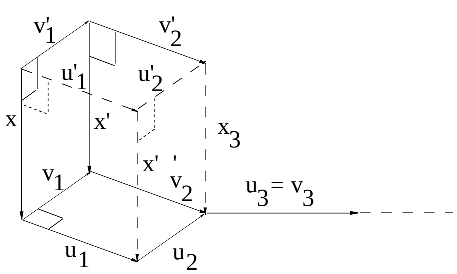
nests u_1. So x does not nest u'_1, and vice versa since d is standard, u'_1 does not nest x. Since $x \perp u'_1$, there exists a unique residual x' of x by u'_1. If u'_1 was nesting v'_1, it would nest x, therefore by axiom 1 and 2 one obtains that there exists a unique residual u'_2 of v'_1 by u'_1 and $u'_2 < x'$: since there is no conflict in d, the case $(k=0, i=1, l=1)$ cannot refute our lemma.



if $l = 2$ then there exists v'_1, v'_2 and x' such that $v'_1 \perp x$, $v'_1[\![x]\!]v_1$, $x[\![v'_1]\!]x'$, $v'_2 < x'$ and $v'_2[\![x']\!]v_2$. Since $u_1 \perp v_1$ and $u_1[\![v_1]\!]v_2$, by axiom 4 there exists a redex u'_1 such that $u'_1[\![v'_1]\!]v'_2$ and $u'_1[\![x]\!]u_1$. By axiom 2, $u'_1 < x$, this proving that the case $(k=0, i=1, l=2)$ cannot refute the lemma.



if $l > 2$ then there exists v'_1, v'_2, x' and x_3 such that $v'_1 \perp x$, $v'_1[\![x]\!]v_1$, $x[\![v'_1]\!]x'$, $v'_2 \perp x'$, $v'_2[\![x']\!]v_2$ and $x'[\![v'_2]\!]x_3$. By axiom 4, there exists a redex u'_1 such that $u'_1[\![v'_1]\!]v'_2$ and $u'_1[\![x]\!]u_1$. By axiom 2, $u'_1 \perp x$ and there exists a unique residual x'' of x by u'_1. By the axiom 2, $u'_1 \perp v'_1$ and there exists a unique residual u'_2 of v'_1 by u'_1. By the axiom 2, $x'' \perp u'_2$ and by finite development, x_3 is the unique residual of x'' by u'_2. Therefore there is a square conflict between x and u_l into d. This being impossible implies that the case $(k=0, i=1, l>2)$ cannot refute our lemma.



□

Proof of Theorem 2

Lemma 2 *Let $d = u_1 \dots u_n$ be a standard derivation. If $z[\![d]\!] = \emptyset$ then $d[\![z]\!]$ is a standard derivation.*

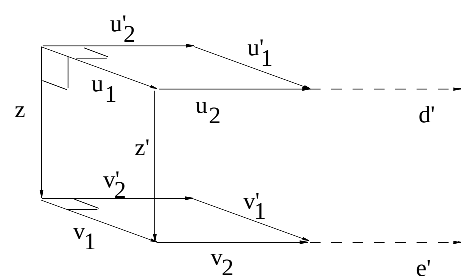
Proof: We will proceed by induction on the length of d. By Proposition 1 and Axiom 1, for any i u_i cannot be nested in a residual of z by $u_1 \dots u_{i-1}$, hence $d[\![z]\!]$ is a derivation of length at most n. It follows that the induction hypothesis can be strengthened to:

> *For all standard derivations d' shorter than d, if Z is a family of redexes such that $Z[\![d']\!] = \emptyset$, $d'[\![Z]\!]$ is standard.*

which is easily proved by induction on the development of Z.

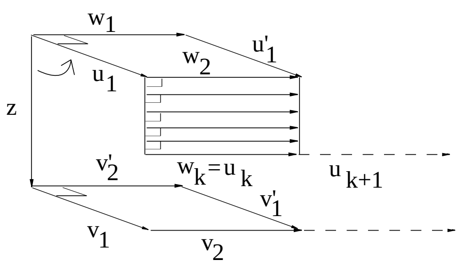
If $d[\![z]\!]$ has length 1 it is trivially standard; if not, let $d = u_1 u_2 d'$ and $d[\![z]\!] = v_1 v_2 e'$. We consider the possible relations between u_1 and z. If $z = u_1$ then $d[\![z]\!] = u_2 d'$ is standard. Otherwise, $z < u_1$ is forbidden by Proposition 1 and Axiom 1, so $u_1[\![z]\!] = \{v_1\}$. Since $u_2 d'$ is shorter than d, we can apply our strengthened induction hypothesis and deduce that $v_2 e' = u_2 d'[\![z[\![u_1]\!]]\!]$ is standard. So all we need to show is that v_1 does not conflict with the rest of $d[\![z]\!]$. This is obviously true if $v1$ creates v_2 or if $[\![v_1]\!]v_2 < v_1$. Hence assume there is $v'_2 \triangleq [\![v_1]\!]v_2 \not\leq v_1$.

If $z \perp u_1$, then z has a unique residual z' by u_1, and $z' \not< u_2$ by Proposition 1. If $z' = u_2$ then $z v_1 v_2 e'$ is a square permutation of d, hence is standard by Proposition 3. Otherwise we can apply Axiom 4 and deduce the existence of a $u'_2 = [\![u_1]\!]u_2 = [\![z]\!]v'_2$. Moreover $u'_2 \not\leq u_1$ since d is standard, and we assumed $v'_2 \not\geq v_1$, so by Axiom 2 we have both $u'_2 \perp u_1$ and $v'_2 \perp v_1$. Let u'_1 and v'_1 be the unique residuals of u_1 and v_1 by u'_2 and v'_2, respectively. By Axiom 2, $z' \not\leq u_2$ implies $z \not\leq u'_2$,m so $u'_2[\![z]\!] = \{v'_2\}$. Hence by Axiom 0 $u'_1[\![z[\![u'_2]\!]]\!] = u'_1[\![z v'_2]\!] = \{v'_1\}$. Also $d'[\![z[\![u'_1 u'_2]\!]]\!] = d'[\![z[\![u_1 u_2]\!]]\!] = e'$, so that $u'_1 d'[\![z[\![u'_2]\!]]\!] = v'_1 e'$. By Proposition 3 $u'_2 u'_1 d'$ is standard, so applying the induction hypothesis to $u'_1 d'$ and $z[\![u'_2]\!]$ we get that $v'_1 e'$ is standard too. Since any conflict from v_1 in $v_1 v_2 e'$ would have to go through v'_1, it follows that $d[\![z]\!]$ is standard.



Finally, we have the $z > u_1$ case. Here Axiom 3 implies that z cannot create $v'_2 \not\geq v_1$, so there is a

$w_1 \triangleq [\![z]\!]v_2'$. By construction of $v_1 v_2 e'$, the first index k such that u_k is not a residual of z exists, and verifies $u_k[\![z[\![u_1 \ldots u_k - 1]\!]]\!] = v_2$. Set for every $1 \leq i < k$, $w_{i+1} = w_i[\![u_i]\!]$. Each w_i is a single redex, and for each $i < k$ we have $u_i \not\leq w_i$. For $i = 1$ this follows from $v_1 \not\leq v_2'$. Since $z > u_1$ by transitivity this implies that $z \not\leq w_1$, and all $i > 1$ cases follow from Axiom 2. Since d is standard we also have $u_{k-1} \not> w_{k-1}$. If $k > 2$ this implies by Axiom 2 that $u_i \not> w_i$ for all $1 < i < k$, and that $z \not> w_1$, which gives $u_1 \not> w_1$ by transitivity with $z > u_1$. In all cases we have $u_i \perp w_i$ for all $i < k$, and $v_1 \perp v_2'$ by Axiom 2. Now we can define $u_i' = u_i[\![w_i]\!]$, $v_1' = v_1[\![v_2']\!]$, and finish the argument as in the $u_1 \perp z$ case: $w_1 u_1' \ldots u_{k-1}' u_k \ldots u_n$ is a square permutation of d, hence is standard by Proposition 3, its residual by z is $v_2' v_1' e'$, hence by induction $v_1' e'$ is standard, and $d[\![z]\!]$ is standard.



□

Proof of Theorem 2: Let $d = u_1 \ldots u_n$ and $e = v_1 \ldots v_{n'}$ be two standard derivations in the same equivalence class w.r.t to general permutations. First observe that it is a basic property of equivalence by permutation that $u_1[\![e]\!] = \emptyset$ and $v_1[\![d]\!] = \emptyset$. By Proposition 1 these imply that either $u_1 = v_1$ or $u_1 \perp v_1$. Using this, we show that $u_1(e[\![u_1]\!]) \simeq_\perp e$. Consider the largest integer k such that for all $0 < i \leq k$, there is a unique residual w_i of u_1 by $v_1 \ldots v_{i-1}$ and $w_i \perp v_i$ if $i < k$. Since $u_1[\![e]\!] = \emptyset$, k is clearly bounded. Now w_k is the first redex in the derivation $e[\![v_1 \ldots v_{k-1}]\!]$, which is permutation equivalent to $v_k \ldots v_{n'}$, and is standard by Lemma 2. By the above observation we must thus have $w_k = v_k$, so $u_1(e[\![u_1]\!]) \simeq_\perp e$. By Proposition 3, $u_1(e[\![u_1]\!])$ is standard, hence $e[\![u_1]\!]$ and $u_2 \ldots u_n$ are shorter equivalent standard derivations. By induction on n they are square permutation equivalent, hence d and e are too. □

6 Abstract systems with incompatibility.

Permutations of reductions have been considered even in the presence of conflicts (critical pairs) in [5]. These TRS are incredibly more difficult to treat. The main reason is that the Church-Rosser property no longer holds. Take $\{A \to B, A \to C\}$. This system has two trivial overlapping rules, and is not Church-Rosser. However, finite developments are still correct, if one considers a set of non overlapping redexes. So, in the setting of abstract reduction systems, it is necessary to introduce a new relation between outgoing arcs of a given term: the incompatibility relation $\#$. We also write $u \uparrow v$ for two compatible redexes.

First, three relations are needed between $\uparrow$ and $\leq$.

Relation 1 $t \# u \Longrightarrow \nexists t'.\ t[\![u]\!]t'$

Relation 2 $t \leq u \Longrightarrow t \uparrow u$

Relation 3 $(t[\![u]\!]t',\ v[\![u]\!]v') \Longrightarrow (t \uparrow v \Leftrightarrow t' \uparrow v')$

Then the 4 previous axioms have to be modified.

Axiom 1 (Linearity) $(u \not\leq t,\ t \uparrow u) \Longrightarrow \exists! t'.\ t[\![u]\!]t'$

Axiom 2 (Context free)
$(u \not\leq t,\ t[\![u]\!]t',\ v[\![u]\!]v') \Longrightarrow (t \Re v \Leftrightarrow t' \Re v')$ *where* $\Re$ *is* $<$ *or* $>$.

Axiom 3 (Creation)
$(t < u,\ t[\![u]\!]t',\ t' \uparrow v,\ \nexists v'.\ v'[\![u]\!]v) \Longrightarrow t' < v$

Axiom 4 (Stability)
$(t \uparrow u,\ t \perp u,\ t[\![u]\!]t',\ u[\![t]\!]u',\ v_1[\![u']\!]v',\ v_2[\![t']\!]v')$
$\Longrightarrow \exists v.\ (v[\![t]\!]v_1,\ v[\![u]\!]v_2)$

Standard reductions are defined just as they were with no critical pairs. The proofs for existence and property 3 are exactly the same in both frameworks since they proceed only by properties on local permutations. The proposition 1 still holds, but such a non-overlapped redex might have no residual, since a redex in the derivation can be incompatible with its residual. And the standard derivations do not verify lemma 2 anymore, in fact the proposition is only true when any residual of z is compatible with the redexes computed in the derivation.

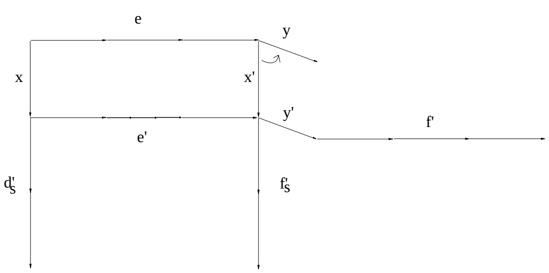
Nevertheless, the uniqueness can be showed by proving that there exists a redex which is reduced but whose residuals are never overlapped by a derivation equivalent to d. (therefore this redex is compatible with any equivalent derivation).

We will sketch this proof of unicity in this new framework: At first, one extends the notion of compatibility on the derivations, and then proceeds by induction on the length of a standard derivation in the permutation class. If a redex is a standard, by the axiom 3 and the finite developments the only standard derivation in its class is itself.

Let $d = xd'$ be standard. We will prove that no redex in a derivation equivalent to d nests a residual of x. Suppose there exists a derivation e equivalent to d such that one of its computed redex nests a residual from x. Hence there exists a sequence $d = d_1, \ldots, d_n$ with d_{i+1} obtained from d_i by a general permutation, such that no residual of x is ever nested into one of these derivations, and that $d_n = ex'y'f$, and y' conflicting with x', the only residual of x by e. This conflict means that there exists y such that $y < x'$ and $y[\![x']\!]y'$.

Since any residual of x is compatible with the one reduced by e, one can choose a derivation $e' \propto e[\![x]\!]$ compute **STD** on $y'f$ and in the same time compute **STD** on $e'y'f$. Let f'_s and d'_s be the results. Since d'_s is square equivalent to d' by induction xd'_s is standard, and $e[\![xd'_s]\!] = \emptyset$, the two derivations being compatible by construction. Hence $x'f_s$ is standard by lemma 2, and $y < x'$. By the property 1, there should be a residual of y by $x'f'_s$, but there is not since $y[\![x'y'f']\!] = \emptyset$.

Hence x is never nested before being computed into any derivation equivalent to d (and therefore cannot be incompatible to an equivalent derivation). Now, suppose there are two standard derivation in the general permutation class, for instance d and e. Then $d[\![x]\!]$ and $e[\![x]\!]$ are easily defined, equivalent, and are both standard. So they are square equivalent, and therefore $d \simeq_\perp e$.



□

7 Examples

It is straightforward to show that the λ-calculus and orthogonal TRS verify the different axioms (without conflicts). A more interesting fact is to take the top-down and left-to-right ordering in λ-terms for $\leq$. It satisfies axioms 1–4. As it is a total ordering, $\perp$ is the empty relation, which means that the square equivalence is the identity. Thus, it is obvious that there is a unique standard reduction in each permutation class. Similarly, the case of strongly sequential orthogonal TRS [2] gives the same result by taking the leftmost of the "sequential indexes".

Instead of merely considering pure λ-terms, we may also examine typed versions, with δ-rules, such as PCF.

As for dags, the nesting relation has to be carefully designed since several paths may lead to a given expression. However, we say that $t \leq u$ if there is a path from t to u in the corresponding dag. Then checking that orthogonal dag systems fulfill axioms 1–4 is not difficult.

Finally, the case of interaction networks [13] is also interesting, since it deals with graphs. There is not a single root, but the non-cyclic property enables us to define nesting. It can be used for proof networks of linear logic.

8 Conclusion

An axiomatic version of the standardisation has been presented. It shows the necessary basic properties between nesting of redexes and residuals. Similarly, one may hope that an analogous treatment of the sequentiality property is feasible, although a priori more difficult because of the uniformity of this property. Thinking of sequentiality in terms of transitions and redexes would shed a new light at an operational level, instead of always thinking of this property in terms of denotational semantics.

Acknowledgements

We thank Ian Jacobs and Chet Murthy for their careful reading.

References

[1] H. B. Curry and R. Feys, "Combinatory logic," *North Holland*, vol. 1, 1958.

[2] G. Huet and J.-J. Lévy, "Computations in orthogonal rewriting systems," in *Computational Logic; Essays in Honor of Alan Robinson* (J.-L. Lassez and G. D. Plotkin, eds.), pp. 394–443, MIT Press, 1991.

[3] R. Hindley, "An abstract form of the Church Rosser theorem," *J. of Symbolic logic*, vol. 34, Dec. 1969.

[4] J. W. Klop, "Combinatory reduction systems," tech. rep., CWI, 1980. Math. Center Tracts 129, Amsterdam.

[5] G. Boudol, "Computational semantics of term rewriting systems," in *Algebraic methods in semantics* (M. Nivat and J. Reynolds, eds.), Cambridge Univ. Press, 1985.

[6] J.-J. Lévy, *Réductions correctes et optimales dans le λ-calcul.* PhD thesis, Univ. of Paris7, Jan. 1978.

[7] G. D. Plotkin, "Church-Rosser categories." unpublished manuscript, 1978.

[8] E. W. Stark, "Concurrent transition systems," *J. Theoretical Computer Science*, vol. 64, May 1989.

[9] M. J. O'Donnell, *Computing in Systems described by Equations.* PhD thesis, Cornell, 1977.

[10] G. Berry and J.-J. Lévy, "Minimal and optimal computations of recursive programs," *JACM*, Jan. 1979.

[11] H. P. Barendregt, *The Lambda Calculus: Its Syntax and Semantics.* North Holland, 1985.

[12] G. Berry, *Modèles complètement adéquats et stables des λ-calculs typés.* PhD thesis, Univ. of Paris7, Mar. 1979.

[13] Y. Lafont, "Interaction nets," in *Seventeenth Annual ACM Symposium on Principles of Programming Languages*, pp. 95–108, ACM, Jan. 1990.