Coefficients of the adapted Chebyshev series

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Abstract

In the report "Formalizing SML's arithmetic by tracking cumulated error bounds" we gave a method for adapting some Chebyshev expansions for use in polynomial approximation when relative error is to be minimised. Here we give a short proof that equation (3.27) correctly defines the adapted coefficients. In particular we show that:

$$\sum_{n=0}^{\infty} a_n T_{2n+1}(x) = x \sum_{n=0}^{\infty} b_n T_{2n}(x)$$

where:

$$b_n \stackrel{\text{def}}{=} (2 - \delta_{0n}) \sum_{i=0}^{\infty} (-)^i a_{n+i}$$

We substitute the definition of the b_n's into the RHS obtaining a double series, and rearrange it into a singly infinite sum by grouping the summands along diagonals:

$$\text{RHS} = x \sum_{n=0}^{\infty} (2 - \delta_{0n}) \sum_{i=0}^{\infty} (-)^i a_{n+i} T_{2n}(x) = x \sum_{k=0}^{\infty} a_k \sum_{n=0}^{k} (-)^{k-n} (2 - \delta_{0n}) T_{2n}(x)$$

Then we prove the equality by considering the multiplier of each a_k on the left and right hand sides. On the LHS the multiplier is clearly $T_{2k+1}(x)$, whereas on the RHS it is $x \sum_{n=0}^{k} (-)^{k-n} (2 - \delta_{0n}) T_{2n}(x)$. We use the recurrence relations (3.16) to show that these two forms are in fact equal:

$$T_{2k+1}(x) \quad = \quad 2x T_{2k}(x) - T_{2k-1}(x)$$

$$
\begin{aligned}
&= 2x(T_{2k}(x) - T_{2k-2}(x)) + T_{2k-3}(x) \\
&= \cdots \\
&= 2x\Big(\sum_{n=1}^{k}(-)^{k-n}T_{2n}(x)\Big) + (-)^k T_1(x) \\
&= x\sum_{n=0}^{k}(-)^{k-n}(2-\delta_{0n})T_{2n}(x)
\end{aligned}
$$

Thus the equality is proved.